A.I. Malcev

Algebraic Systems

The book gives a systematic presentation of the general theory of algebraic systems. Algebraic systems (sets with operations and relations defined on them) are among the most fundamental of algebraic structures. The theory was developed primarily during the past twenty years bordering on classical general algebra and mathematical logic though some of its basic ideas go back to the 1930's. A most important feature of the theory is the connection between the general properties of classes of algebraic systems and the syntactic characteristics of the language on which the classes are defined. This makes it possible to penetrate very deeply into the structure of many classical algebras and geometrical theories. At the same time, some of the subtler distinctive features of the predicate calculus, the formal language most used today, are brought to light. Some of the main topics considered in the book are classical algebras, first and second order languages, products and complete classes, quasivarieties and varieties the latter including a number of the author's own results.

A. I. Malcev

Algebraic Systems

A. I. Malcev

Algebraic Systems

Translated from the Russian by

B. D. Seckler and A. P. Doohovskoy

With 25 Figures

Akademie-Verlag · Berlin

1973

А. И. Мальцев
Алгебраические системы
Verlag Nauka, Moskau

Erschienen im Akademie-Verlag GmbH, 108 Berlin, Leipziger Str. 3 – 4
© by Springer-Verlag, Berlin · Heidelberg 1973
Lizenznummer: 202 · 100/540/73
Einband: Helga Klein
Gesamtherstellung: VEB Druckhaus „Maxim Gorki", 74 Altenburg
Bestellnummer: 761 692 9 (5991) · ES 19 B 1, 19 B 2
Printed in German Democratic Republic

From the Editors

The author of the book is the eminent Soviet Academician Anatolii Ivanovič Mal'cev whose untimely death occurred July 7, 1967 in his fifty-eight year.

The scientific legacy of A. I. Mal'cev is exceptionally rich and extensive. Many fundamental results in groups, rings, linear algebra, algebraic topology, Lie groups and mathematical logic are due to him.

A. I. Mal'cev was also one of the originators of the theory of algebraic systems. It came into being when the techniques of mathematical logic began to be applied to algebra and so it occupies a position bordering on algebra and logic. Mal'cev published a long series of articles devoted to algebraic systems and delivered several brilliant review talks on the subject at the Third and Fourth All-Union Mathematics Conferences [43], [45]. In a report to the 1966 International Congress of Mathematicians in Moscow, he set forth a number of new ideas relating to the theory of algebraic systems [48].

Mal'cev began work on the book *Algebraic Systems* in 1951 during his stay in Ivanov. In 1953, a typewritten copy of the first part was submitted to the Mathematics Department of the Ivanov Pedagogical Institute. Since then, many Soviet algebraists have availed themselves of it. However further work on the book was suspended and it was not resumed again until 1964 in Novosibirsk. Since the theory of algebraic systems had by that time been supplemented by a great number of new results that had altered its character essentially, a new book was actually begun. Mal'cev had contemplated writing a two-volume work to be entitled *General Algebra*. The first volume devoted to basic structures, predicate calculi and varieties was included in the production plans of the Physics-Mathematics Literature Press for 1967. It was not completed in time and its publication was postponed. But Mal'cev continued working on the book and only his sudden death prevented its final completion.

Not having any information regarding the content of the second volume at our disposal, we decided to retain the original title of *Algebraic Systems*, especially since the manuscript left behind was so named.

A typewritten copy of the book had been sent to us by Mal'cev together with several versions of a proposed table of contents. We

naturally chose that version which most closely corresponded to the actual content of the manuscript. In accordance with the author's plans, we incorporated the chapter *Classical Algebras* from the Ivanov version of the manuscript (appropriately edited and augmented with certain material) as well as necessary information on cardinal and ordinal numbers from Hausdorff's book [18]. We also added a few other missing connective links. The latter additions have been starred in the table of contents. The first chapter also underwent some editorial revision since it had only considered algebraic systems of finite signature whereas the subsequent chapters assume an arbitrary signature.

To handle the book, the reader should be conversant with the material in standard university courses in modern algebra and mathematical logic. Only in isolated places is reference made to special literature.

April 1968

D. Smirnov and M. Taiclin

Author's Preface

As far back as the 1920's, algebra had been accepted as the science studying the properties of sets on which there is defined a particular system of operations. However up until the forties the overwhelming majority of algebraists were investigating merely a few kinds of algebraic structures. These were primarily groups, rings and lattices. The first general theoretical work dealing with arbitrary sets with arbitrary operations is due to G. Birkhoff (1935). During these same years, A. Tarski published an important paper in which he formulated the basic principles of a theory of sets equipped with a system of relations. Such sets are now called models. In contrast to algebra, model theory made abundant use of the apparatus of mathematical logic. The possibility of making fruitful use of logic not only to study universal algebras but also the more classical parts of algebra such as group theory was discovered by the author in 1936.

During the next twenty-five years, it gradually became clear that the theory of universal algebras and model theory are very intimately related despite a certain difference in the nature of their problems. And it is therefore meaningful to speak of a single theory of algebraic systems dealing with sets on which there is defined a series of operations and relations (algebraic systems). The formal apparatus of the theory is the language of the so-called applied predicate calculus. Thus the theory can be considered to border on logic and algebra.

The author attempted to justify this viewpoint in his reports to the All-Union Conferences in 1956 and 1961.

Starting in 1951, the author had repeated occasion to give special lecture courses devoted to the basic concepts and certain portions of the theory of algebraic systems. In 1953—54, notes were written for one of these courses and several dozens of typewritten copies were hand distributed in various cities. But their preparation for publication was not completed in time. In 1964, the author once more found himself obliged to give a course in algebraic systems and a group of students and colleagues at Novosibirsk University suggested that the old notes be rotaprinted. But the state of the theory had changed so drastically during the preceding ten years that revision of the notes became necessary. This led to the creation of the present book.

The content of the book roughly corresponds to the content of the author's two review reports mentioned above.

Contents

Chapter I

General Concepts

1. Relations and Mappings 1

1.1 Sets 1
1.2 Relations 8
1.3 Mappings 11
1.4 Equivalence Relations 13
1.5 Partial and Linear Orderings 19
1.6 Multivalued and Partial Mappings 21
1.7 Cardinal and Ordinal Numbers 24
Problems and Complements 28

2. Models and Algebras 29

2.1 n-ary Relations and Functions 29
2.2 Algebraic Systems 32
2.3 Subsystems. Generating Sets 37
2.4 Congruences 43
2.5 Direct Products 52
2.6 Operations on Cardinal and Ordinal Numbers 62
Problems and Complements 66

Chapter II

Classical Algebras

3. Groupoids and Groups 67

3.1 Groupoids and Semigroups 67
3.2 Quasigroups and Loops 72
3.3 Groups 74
Problems and Complements 80

4. Rings and Fields 81

4.1 Rings 81
4.2 Algebraically Closed Fields 87
4.3 Alternative Skew Fields 91
4.4 Linear Algebras 94
Problems and Complements 100

5. Lattices (Structures) 100

5.1 Lattices 100
5.2 Modular and Distributive Lattices. Boolean Algebras 104

Chapter III

First and Second-Order Languages

6. Syntax and Semantics 109
6.1 Terms 109
6.2 Formulas 115
6.3 Second-Order Properties 121
6.4 Elementary Theories and Axiomatic Classes 126
Problems and Complements 128

7. Classification of Formulas 129
7.1 ∀-Formulas and ∃-Formulas 129
7.2 Universall Axiomatic Subclasses 135
7.3 ∀∃-Formulas and ∃∀-Formulas 140
7.4 Positive Formulas 143
7.5 Multiplicatively Stable Formulas 145

Chapter IV

Products and Complete Classes

8. Filters and Filtered Products 153
8.1 Filters and Ultrafilters 153
8.2 Ultraproducts 156
8.3 Applications of Ultraproducts 163
8.4 Conditionally Filtered Formulas 168
8.5 Cardinality of Ultraproducts 172
8.6 Regular Products 177
Problems and Complements 184

9. Indistinguishability and Elementary Embedding 185
9.1 Elementary Embedding 185
9.2 Elementary Subsystems 191

10. Completeness and Model Completeness 195
10.1 Complete Systems of Formulas 195
10.2 Model Completeness 201
Problems and Complements 209

Chapter V

Quasivarieties

11. General Properties 210
11.1 Characteristic Properties 210
11.2 Defining Relations 217
11.3 Replicas 228
Problems and Complements 236

12. Free Systems and Free Compositions 236
12.1 Free Compositions 236
12.2 Independent Elements and Free Systems 247
12.3 Amalgamated Compositions 255
Problems and Complements 266

Chapter VI
Varieties

13. General Properties 267
13.1 Structural Characteristics 267
13.2 Ranks of Varieties 271
13.3 Varieties of Unoids 276
Problems and Complements 282
14. Primitive Closures 282
14.1 Generating Systems 283
14.2 The Lattice of Varieties 289
14.3 Minimal Varieties and Quasivarieties 295
Problems and Complements 303

Bibliography 305

Name Index 309

Subject Index 310

Special Notation and Symbols

A', $\bar{A}$	complement of A 3
$\|A\|$	cardinality of A 24
$A\alpha$	range of α 21
A^n	cartesian n-th power of A 29
$a\alpha$, a^α	image of a under α 11
Aut G	automorphism group 78
$B\alpha^{-1}$	domain of α 21
$D(\mathfrak{A})$	diagram of model A 136
$D^\wedge(\mathfrak{A})$	conjunction of formulas in $D(\mathfrak{A})$ 137
Da $(\mathfrak{A})$	atomic diagram of $\mathfrak{A}$ 230
df	denotes the formula 126
$\mathscr{D}_Y$ $(\mathfrak{A})$	Y-description of $\mathfrak{A}$ 189
F	is false; falsity 30
$F\mathfrak{K}$	class of systems isomorphic to filtered products 215
$F_m^{(n)}$	operator symbol 111
F_ξ	basic operation 32
$H\mathfrak{L}$	class of all homomorphic images of $\mathfrak{L}$-systems 249
$I\mathfrak{K}$	set of all identities of given signature true in all $\mathfrak{K}$-systems 268
$I_n\mathfrak{K}$	set of all identities of rank $\leqq n$ true in $\mathfrak{K}$ 272
$K\mathfrak{S}$	class of systems in which formulas of $\mathfrak{S}$ are true 128
$K^{\{\mathfrak{A}_a \mid a\in I\}}_{\mathscr{A}}$	set of indices α for which $\mathscr{A}$ is true in $\mathfrak{A}_\alpha$ 178
$L\mathfrak{K}$	class of models locally embeddable in $\mathfrak{K}$ 138
$l(X)$	length of word X 109
$M(\mathfrak{M})$	subvarieties of variety $\mathfrak{M}$ 290
N	$\{0, 1, 2, \ldots\}$ 2
$n(Q)$	arity of Q 29
$o(A)$	order type of A 25
P_a	initial segment determined by a 26
$\mathscr{P}(A)$	set of all subsets of A 27
$P_m^{(n)}$	predicate symbol 111
P_η	basic predicate 32
$\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i$	complete regular product 179
$q_a(\mathfrak{K})$	axiomatic quasirank of $\mathfrak{K}$ 275
$q_b(\mathfrak{K})$	basis quasirank of $\mathfrak{K}$ 275
$Q\mathfrak{K}$	set of quasi-identities true in $\mathfrak{K}$ 215
$Q_n\mathfrak{K}$	set of quasi-identities of rank n true in $\mathfrak{K}$ 275
$r_a(\mathfrak{K})$	axiomatic rank of $\mathfrak{K}$ 271
$R\mathfrak{K}$	replica closure 232
$S\|_A\|$, S_A	symmetric (semi) group of A 67, 78
$S\mathfrak{L}$	class of all subsystems of $\mathfrak{L}$-systems 215
T	is true; truth 30
Th $\mathfrak{K}$	elementary theory for $\mathfrak{K}$ 128
$W(\alpha)$	set of ordinals less than α 26
α^e	equivalence closure of α 18
$\alpha\beta$	composition of relations 9
α^n	n-fold composition of α 10
α^{-1}	relation inverse to α 9
$\prod\mathfrak{K}$ ·	class of isomorphic copies of direct products 232
$\prod_{a\in A}\mathfrak{A}_a$	direct product 53
$\prod^*_{\mathfrak{B}}\mathfrak{A}_i$	$\mathfrak{K}$-composition of systems $\mathfrak{A}_i$ 237
σ-class	equivalence class under σ 14
τ	type of a system 32
Ω_P, Ω_F	predicate, operator domain 32
Ω_M	domain of representing model 33
$\mathfrak{S}_{m,n}$	variety of algebras 303
$\forall$	universal quantifier 111
$\forall^n$	n-fold universal quantifier 139
$\exists$	existential quantifier 111
ι; (ι_A)	identity relation 10
$\&\ \vee\ \neg\ \rightarrow$	logical symbols 111
$\aleph_0$	cardinality of countable sets 24
$[a]$, $[a]_\sigma$	equivalence class containing a 14, 44
A/σ	quotient set modulo σ 14, 22

$\langle A, \{P_\eta\}\rangle$ model $\mathfrak{A}$ 32
$\langle A, \Omega_M\rangle$ representing model 33
$\langle A, \Omega_F, \Omega_P\rangle$ algebraic system 32
$(a_1, \ldots, a_n)\alpha$ value of function α 29
$\langle I, D\rangle$ filter D over I 153
$[x, y]$ commutator of x and y 79
$[x, y, z]$ associator of x, y and z 83
$\backslash$, $/$ left and right division 72
$x \alpha y$ x in α-relation to y 8
$\alpha \textcircled{e} \beta$ equivalence closure of α 18
$a \vee b$, $a \wedge b$ l. u. b. and g. l. b. of a and b 100
$A - B$ difference of A and B 3
$\mathscr{A} \underset{\mathfrak{A}}{=} \mathsf{T}$ formula is true 138

$\mathfrak{A} \cong \mathfrak{B}$ $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic 58, 158
$\mathfrak{A} \equiv \mathfrak{B}$ $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent 185
$\mathfrak{A} \leqq_\Upsilon \mathfrak{B}$ $\mathfrak{A}$ is Υ-indistinguishable from $\mathfrak{B}$ 186
$\mathfrak{A} \equiv_\Upsilon \mathfrak{B}$ $\mathfrak{A}$ and $\mathfrak{B}$ are Υ-equivalent 186
$\mathfrak{A} <_\Upsilon \mathfrak{B}$ $\mathfrak{A}$ is an Υ-submodel of $\mathfrak{B}$ 187
$A \subseteq \mathfrak{B}$ $\mathfrak{A}$ is isomorphic to a subsystem of $\mathfrak{B}$ 215
$\mathfrak{A} = \prod^*_{\mathfrak{C}} \mathfrak{A}_i$ mixed free product over amalgmation 263
$\mathfrak{A} = \mathfrak{F}_m(\mathfrak{K})$ $\mathfrak{A}$ is of free rank in $\mathfrak{K}$ 249

Chapter I

General Concepts

1. Relations and Mappings

1.1 Sets. A collection of objects or concepts united by some common property is called a *set*. For example, one speaks of the set of symbols in a printed line, the set of natural numbers, the set of persons who are currently university students and so on. The objects comprising a set are its *elements*. Throughout the following, sets will be denoted as a rule by capital letters and elements by lower case Latin or Greek letters. The statement that an object a is an element of a set A is written briefly as $a \in A$. The notation $a \notin A$ or $a \bar{\in} A$ signifies that a is not an element of A.

The elements of a set may themselves be sets. For instance, various student groups may be regarded as elements of the set of all groups in a university even though each group is itself a set of students.

Two sets are said to be *equal* if they have exactly the same elements, i.e., if each element of one set is an element of the other. This means that to specify a set completely, one merely has to list all of its elements. Therefore, wherever convenient, a set will be defined by writing its elements down within braces in any order. Thus, $\{4, 1, 3\}$ is the set consisting of the numbers 1, 3 and 4. The sets $\{1, 3, 1, 5\}$ and $\{1,3,5\}$ are to be viewed as equal since listing 1 twice among the elements is of no import in the defining of the set.

One often has to consider sets containing a single element. The set $\{a\}$, *singleton* a, consisting of a single element a and the element a are distinct entities. In particular, one has to distinguish between the element a, the set $\{a\}$ consisting of element a, the set $\{\{a\}\}$ consisting of singleton a, etc.

Apart from sets that do contain elements, one also has to deal with the *empty* or *void* set having no element in it. All empty sets are equal by definition and are denoted by the symbol $\varnothing$. Observe that the set $\{\varnothing\}$ has $\varnothing$ as an element and so is not empty.

A set A is said to be a *subset* or *part* of set B, symbolically $A \subseteq B$, if each element of A is an element of B. For example, the set of even natural numbers is a subset of the set of all natural numbers.

By this definition, a set A and the empty set $\varnothing$ are subsets of A.

A nonempty subset of A differing from A is called a *proper* subset of A. The notation $A \subset B$ means that A is a proper subset of B.

The empty set $\varnothing$ has just one subset, itself. The set $\{1\}$ consisting of the one element 1 has two subsets: $\varnothing$ and $\{1\}$. The set $\{1, 2\}$ of two elements has four subsets: $\varnothing$, $\{1\}$, $\{2\}$ and $\{1, 2\}$. It is easy to verify that a set with a finite number of n elements has 2^n distinct subsets.

Let A, B and C be arbitrary sets. If $A \subseteq B$ and $B \subseteq A$, then clearly $A = B$. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. If $A \subset B$ and $B \subseteq C$, then $A \subset C$, etc.

Instead of the terms *set* and *subset*, one often uses *collection* and *subcollection*, *system* and *subsystem*, *family* and *subfamily*, etc. If $A \subseteq B$, one also says that B is a *superset* (*supersystem*) of A.

Let P be some property of objects and let $P(x)$ stand for the fact that object x has property P. Then the set of objects x that possess property P is denoted by $\{x \mid P(x)\}$. For example, let N denote the natural numbers 0, 1, 2, ...*). Then $\{x \mid x > 5, x \in N\}$ is the set of natural numbers greater than 5, $\{x \mid 2 < x < 6, x \in N\}$ is the set $\{3, 4, 5\}$ and $\{x \mid x < 0, x \in N\} = \varnothing$.

The *union* or *sum* of sets A and B, symbolically $A \cup B$, is the set obtained by combining the elements of A and B into a single set. Thus, the statement $a \in A \cup B$ means that $a \in A$ or $a \in B$. For example,

$$\{1, 2, \} \cup \{2, 3, 5, 6\} = \{1, 2, 3, 4, 5, 6\}.$$

In particular, for any set A we have

$$A \cup A = A, \qquad \varnothing \cup A = A.$$

The expression

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

stands for the union of the sets $A_1, A_2, \ldots, A_n$.

The *intersection* or *common part* of sets A and B, symbolically $A \cap B$, is the set containing those and only those objects that belong to A and B simultaneously. If A and B have no elements in common, their intersection is empty. Thus,

$$\{1, 3\} \cap \{2, 4, 5,\} = \varnothing,$$

$$\{1, 2, 5\} \cap \{1, 5, 6\} = \{1, 5\}.$$

In particular, for any set A we have

$$A \cap A = A, \qquad \varnothing \cap A = \varnothing.$$

*) The author's convention of including zero among the natural numbers will be retained throughout [*Trans.*].

The expression

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

denotes the intersection of the family of sets $A_1, A_2, \ldots, A_n$.

The *difference* of sets A and B, symbolically $A - B$, is the collection of elements in A that are not in B. For example,

$$\{0, 1, 2, 3\} - \{1, 3, 5\} = \{0, 2\}.$$

Hence it follows that for any sets A and B we have

$$A - A = \varnothing, \qquad A - \varnothing = A, \qquad \varnothing - A = \varnothing,$$

$$A - B = A - (A \cap B).$$

Sometimes set union is viewed as the analogue of addition of numbers and set difference as the analogue of subtraction of numbers. However, this analogy is not quite complete. For example, if $A \cup B = C$, it does not necessarily follow that $A = C - B$. Indeed, let

$$A = \{0, 1, 4, 5\}, \qquad B = \{1, 2, 3, 5\}.$$

Then we have

$$A \cup B = \{0, 1, 2, 3, 4, 5\},$$

$$\{0, 1, 2, 3, 4, 5\} - B = \{0, 4\} \neq A.$$

A proper conclusion is the following: if $A \cup B = C$ and $A \cap B = \varnothing$, then $A = C - B$.

If $A \subseteq B$, the difference $B - A$ is called the *complement* of A in B. Whenever the set B is fixed in advance, one writes briefly A' instead of $B - A$ (sometimes $\bar{A}$) and one speaks simply of the *complement* of A. Thus if the basic set B is the set N of natural numbers, then the complement of the collection of odd numbers is the collection of even numbers. The complement of the collection of numbers each divisible by an odd number greater than one is the collection of powers of two: $2^0, 2^1, 2^2, 2^3, \ldots$

The set-theoretic operations of union, intersection and complementation have a very obvious graphic interpretation. As basic set, take the collection of points in the plane. Let A be a circle in the plane, i.e., the set of points lying within and on its circumference (see Fig. 1). Then A' is the set of points exterior to the circumference (in Fig. 1, the hatched region).

Consider now two circles A and B. Fig. 2 depicts their union $A \cup B$ (the hatched region), while Fig. 3 depicts the set $A \cap B$ (the doubly hatched region). The shaded region in Fig. 4 represents $A - B$.

The arithmetic operations of addition and multiplication of natural numbers are operations on pairs of numbers, or as one says, binary

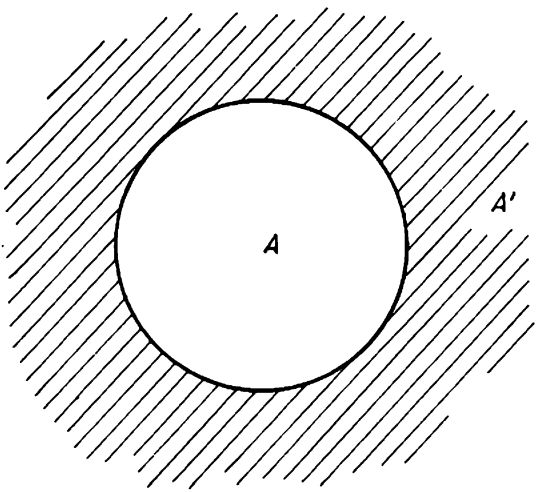

Fig. 1

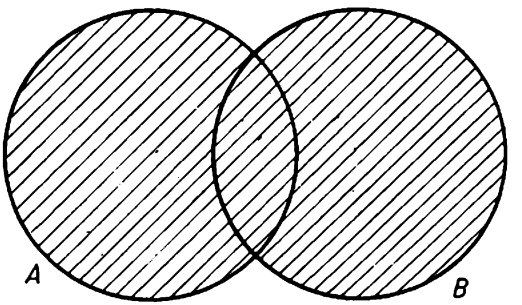

Fig. 2

operations. The set-theoretic operations of union and intersection are good examples of *binary* set operations, i.e., operations on objects of a still simpler logical nature than the natural numbers. The squaring of a number and set complementation are operations on a single object. Such operations are called *unary*.

Let U be a fixed set and let $\mathfrak{U}$ denote the collection of all subsets of U. The operations of union, intersection and complementation (with

respect to U) are applicable to the elements of $\mathfrak{U}$. The collection $\mathfrak{U}$ is closed with respect to the three operations in the sense that their application to the elements of $\mathfrak{U}$ results in elements belonging to $\mathfrak{U}$. The system consisting of the collection $\mathfrak{U}$ and the operations $\cup$, $\cap$, and $'$ is called the *Boolean algebra of subsets* of U.

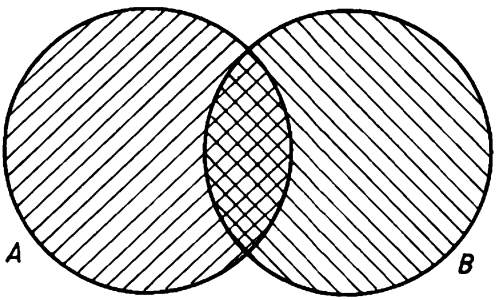

Fig. 3

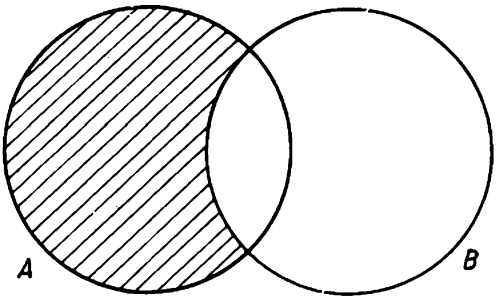

Fig. 4

The arithmetic operations of addition and multiplication are related by the familiar associative, commutative and distributive laws. The Boolean operations of union, intersection and complementation obey similar laws as well as a series of others. We shall now write down a few of the main laws since their abstract versions (see Section 5.2) play an important role in many parts of algebra and logic:

B 1) $x \cup x = x$;
B 2) $x \cup y = y \cup x$;
B 3) $(x \cup y) \cup z = x \cup (y \cup z)$;
B 4) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$;
B 5) $(x')' = x$;
B 6) $(x \cup y)' = x' \cap y'$;
B 7) $x \cup (y \cap y') = x$.

These identities can easily be shown to hold graphically or by use of simple reasoning. For example, it is immediately apparent from Fig. 5 that if x, y and z are any sets, the left and right-hand sides of B 4 represent the same shaded portion of the plane.

Relations B 1 through B 7 are identities. Therefore the replacement of x, y and z by x', y' and z' in each one of them again yields an identity. These new identities can then be transformed with the help of both the old

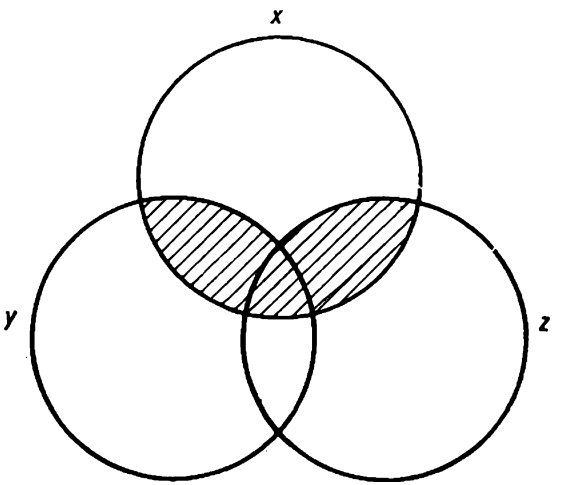

Fig. 5

ones and new ones, etc. As a result, we obtain from B 1—B 7 a number of identities called *formal consequences* of the given ones. This process of deriving new identities from the given ones will be analyzed in all its details later on in Section 11.2. Here we shall merely show that identities B1—B7 lead in a formal way to the following identities in which $x'' = (x')'$:

$B_0 1)$ $x \cap x = x$;
$B_0 2)$ $x \cap y = y \cap x$;
$B_0 3)$ $(x \cap y) \cap z = x \cap (y \cap z)$;
$B_0 4)$ $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$;
$B_0 5)$ $x'' = x$;
$B_0 6)$ $(x \cap y)' = x' \cup y'$;
$B_0 7)$ $x \cap (y \cup y') = x$.

They are all obtained in a similar way. We shall only derive from B1—B7 the identity $B_0 6$. Replacing x and y in B6 by x' and y', we

obtain $(x' \cup y')' = x'' \cap y''$. In view of B5, this last relation may be represented in the form $(x' \cup y')' = x \cap y$. Taking complements of both sides and again applying B5, we arrive at the required relation $B_0 6$.

From a purely formal aspect, identities $B_0 1 - B_0 7$ can be obtained from B1—B7 by interchange of the roles of the symbols $\cup$ and $\cap$ and they are therefore referred to as the *duals* of the latter identities.

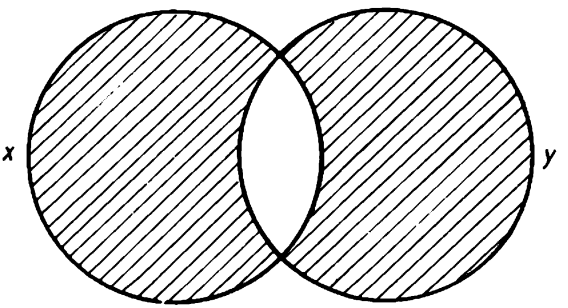

Fig. 6

In parallel with the operation of set difference one sometimes considers the operation of *symmetric difference* defined by

$$x \oplus y = (x - y) \cup (y - x). \tag{1}$$

A graphical representation of the symmetric difference $x \oplus y$ is given in Fig. 6. It is clearly apparent that

$$(x \oplus y) \oplus y = x.$$

Therefore if we introduce a new subtraction (difference) operation $\ominus$ by means of the formula

$$x \ominus y = x \oplus y,$$

then the two operations will be related to each other by the customary arithmetic law,

$$x \oplus y = z \Leftrightarrow y = z \ominus x,$$

wherein $\Leftrightarrow$ symbolizes the word "equivalent". Moreover, it is easy to verify that the usual distributive law holds for the operations $\oplus$ and $\cap$, namely

$$x \cap (y \oplus z) = (x \cap y) \oplus (x \cap z). \tag{2}$$

The system consisting of the collection $\mathfrak{U}$ of all subsets of a set U and operations $\oplus$ and $\cap$ is called a *Boolean ring* of subsets of U. Besides the commutative and associative laws for $\cap$ ($B_0 2$ and $B_0 3$) and the distributive law (2), the following laws are also satisfied in a Boolean ring:

$$x \oplus x = \varnothing, \quad x \oplus \varnothing = x,$$

$$x \oplus y = y \oplus x,$$

$$(x \oplus y) \oplus z = x \oplus (y \oplus z).$$

They are simple consequences of identities B1—B7 and formula (1) defining symmetric difference of sets.

We observe further that in a Boolean algebra the inclusion of subsets: $A \subseteq B$ is equivalent to each of the relations

$$A \cup B = B, \quad A \cap B = A.$$

It is also clear that the inclusion $A \subseteq B$ is equivalent to its dual $B' \subseteq A'$.

It would be natural to ask whether identities B1—B7 and their formal consequences exhaust all identities that relate the operations $\cup$, $\cap$, and $'$ in an arbitrary algebra of subsets. The answer is in the affirmative and this will be demonstrated below (see Sections 5.2 and 7.1).

1.2 Relations. The *cartesian product* of sets A and B, symbolically $A \times B$, is the collection of all pairs of the form (a, b), where $a \in A$ and $b \in B$. For example, if $A = \{1, 2\}$ and $B = \{4, 5\}$, then $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5)\}$. Further,

$$B \times A = \{(4, 1), (4, 2), (5, 1), (5, 2)\},$$

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

If A and B are finite sets containing m and n elements respectively, then the product $A \times B$ contains mn pairs.

Each subset α of a cartesian product $A \times B$ of arbitrary sets A and B is said to be a *relation on the pair A and B* or a *relation from A to B*. If $(a, b) \in \alpha$, then one says that *a stands in relation α to b*, or that α is a true relation for a and b. Instead of $(a, b) \in \alpha$, one also writes $a \alpha b$.

A relation defined on the pair A, A is called a *binary relation in A*.

Since the relations on a fixed pair of sets A and B are subsets of $A \times B$, the collection of all such relations forms a Boolean algebra under union, intersection and complementation of relations. In parti-

cular, for any $a \in A$ and $b \in B$,

$$a(\alpha \cup \beta)b \Leftrightarrow a\alpha b \text{ or } a\beta b,$$

$$a(\alpha \cap \beta)b \Leftrightarrow a\alpha b \text{ and } a\beta b,$$

$$a\alpha' b \Leftrightarrow \text{not } a\alpha b.$$

Therefore instead of union, intersection and complementation of relations, one often speaks of *disjunction*, *conjunction* and *negation*.

For example, the equality relation $(=)$ in the set N of natural numbers can be thought of as the collection of all "diagonal" pairs: $(0, 0)$, $(1, 1)$, ... The complement of this relation is the inequality relation $(\neq)$. The order relation $<$ is the collection of pairs (a, b) for which $a < b$. The relation $\leqq$ is the same as the union $< \cup =$. The intersection $< \cap =$ is empty, i.e., it is a completely false relation. By contrast, the union $\leqq \cup >$ is a completely true relation.

Apart from union, intersection and complementation, there are two more operations on relations that are of great importance. These are inversion and composition (multiplication) of relations defined as follows.

If α is a relation from A to B, then the *inverse* of α (symbolically α^{-1}) is the relation α^{-1} from B to A consisting of the pairs (b, a) for which $(a, b) \in \alpha$. In other words,

$$b\alpha^{-1}a \Leftrightarrow a\alpha b.$$

For example, if $<$ is order relation, then

$$a <^{-1} a \Leftrightarrow b < a \Leftrightarrow a > b,$$

and so $<^{-1} = >$.

Suppose now that α is a relation from A to B and β is a relation from B to C. The *composition* or *product* $\alpha\beta$ *) of α and β is the relation from A to C such that $a(\alpha\beta)c$ $(a \in A,\ c \in C)$ is true if and only if there exists an element x in B for which $a\alpha x$ and $x\beta c$. If α is defined on A and B, β is defined on C and D and $B \neq C$, then the composition $\alpha\beta$ is to be viewed as undefined. The term *composite* will also be used.

Thus in general, the composition of two relations may or may not be defined. The matter changes if one considers the collection of all binary relations in a fixed set A. In that event, the inverse of a relation and the composition of any two relations will again be a binary relation in A.

---

*) Though the notation $\alpha \circ \beta$ is more common in the literature, the original Russian notation will be retained throughout the text [*Trans.*].

Let us consider a few examples. Introduce the notation

$$\alpha\alpha = \alpha^2, \quad \alpha^2\alpha = \alpha^3, \ \ldots \tag{1}$$

Let us see what meaning is conveyed by the relations $<^2$, $<^3$, ... where $<$ is order in the set N of natural numbers. By definition, $a <^2 b$ holds true if and only if there is some natural number x such that $a < x$ and $x < b$. For such an x to exist, it is necessary and sufficient that $a + 1 < b$. Hence,

$$\begin{aligned} a <^2 b &\Leftrightarrow a + 1 < b, \\ a <^3 b &\Leftrightarrow a + 2 < b, \\ &\ldots \end{aligned} \tag{2}$$

Similarly,

$$a(< \cdot >)b \Leftrightarrow \exists x(a < x \,\&\, x > b) \Leftrightarrow \exists x\big(x > \max(a, b)\big). \tag{3}$$

Here $\exists x$ symbolizes the expression *"there exists an x such that"* and & is the conjunction *"and"*. If we interchange the order of the factors $<$ and $>$ in (3), we obtain

$$a(> \cdot <)b \Leftrightarrow \exists x(a > x \,\&\, x < b) \Leftrightarrow \exists x\big(x < \min(a, b)\big). \tag{4}$$

Two binary relations α and β in A are said to be *permutable* if $\alpha\beta = \beta\alpha$, i.e., if the composition of α and β is commutative. Formulas (3) and (4) show that the relations $<$ and $<^{-1} = >$ are not permutable on the natural numbers.

The identity relation in a set A is often denoted by ι_A. It consists of all diagonal pairs (a, a), $a \in A$. If A has been fixed in advance, one writes ι instead of ι_A. It is easy to see that a binary relation α in a set A satisfies

$$\alpha\iota_A = \iota_A\alpha. \tag{5}$$

However, the example in (4) shows that the relation $\alpha\alpha^{-1} = \iota$ need not hold for some relations α.

Starting from the definitions of composition and inversion of relations, one can easily show that the following identities hold for any relations α, β and γ:

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma, \tag{6}$$

$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}. \tag{7}$$

If the relations are defined on pairs of distinct sets, then one must interpret identities (6) and (7) in the following sense. If the left-hand or right-hand side of either equation is well-defined, then the other side is well-defined and the two sides have the same value.

Apart from identities (6) and (7) concerning composition and inversion of relations, there is a series of identities relating these operations to the Boolean operations of union, intersection and complementation. For example, if α, β and γ are any relations on an arbitrary set A, then

$$(\alpha^{-1})' = (\alpha')^{-1}, \tag{8}$$

$$(\alpha \cup \beta)^{-1} = \alpha^{-1} \cup \beta^{-1}, \quad (\alpha \cap \beta)^{-1} = \alpha^{-1} \cap \beta^{-1}, \tag{9}$$

$$\alpha(\beta \cup \gamma) = \alpha\beta \cup \alpha\gamma, \quad (\beta \cup \gamma)\alpha = \beta\alpha \cup \gamma\alpha. \tag{10}$$

Observe that the formulas for intersection analogous to (10) are false.

The system consisting of the collection of all binary relations defined on a set A and the operations $\cup$, $\cap$, $'$, $\circ$, $^{-1}$ on these relations is said to be a *relation algebra* on A. The identities (6)—(10) and the Boolean identities B1—B7 of Section 1.1 exemplify the identities that hold in any relation algebra. The problem of finding a complete system of identities for a relation algebra is more complicated than for a Boolean algebra.

1.3 Mappings. A relation α defined on a pair of sets A and B is called a *mapping* of A into B if to each $a \in A$ there is exactly one element $b \in B$ such that $a\alpha b$. (The terms *function*, *correspondence* and *transformation* are synomymous with mapping.) The element b is called the *image* of a under α and is denoted by a^α or $a\alpha$ If $b = a\alpha$, then a is called the *pre-image* of b under α. The set of all pre-images of b in A under a given mapping α is the *complete pre-image* of b in A.

It is sometimes convenient to write a mapping as a two-row array. The elements of the set A are written down in arbitrary order in the first row and their images in B beneath them. For example, the array

$$\alpha = \begin{pmatrix} 1 & 5 & 3 & 4 \\ 3 & 4 & 1 & 3 \end{pmatrix}$$

defines a mapping of $\{1, 5, 3, 4\}$ into itself under which $1^\alpha = 3$, $5^\alpha = 4$, $3^\alpha = 1$ and $4^\alpha = 3$.

Let α be a mapping of A into B and β a mapping of B into C. The composition $\alpha\beta$ is a mapping of A into C such that for any $x \in A$,

$$x(\alpha\beta) = (x\alpha)\beta.$$

To show this, let $x(\alpha\beta) = c$. For some suitable $y \in B$ we have $x\alpha y$ and $y\beta c$. Hence $x\alpha = y$ and so $c = (x\alpha)\beta$. Conversely, from $c = (x\alpha)\beta$ it follows that $(x\alpha)\beta c$. Since $x\alpha(x^\alpha)$, we have $x(\alpha\beta)c$ or $c = x(\alpha\beta)$.

The way to construct a composition of mappings in array form is immediately evident from the following example:

$$\begin{pmatrix}1 & 2 & 3 & 4\\ a & b & c & d\end{pmatrix}\begin{pmatrix}a & b & c & d\\ x & y & u & v\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ x & y & u & v\end{pmatrix}.$$

A mapping α of A into B is said to be a mapping of A *onto* B if each element $b \in B$ has at least one pre-image in A, in other words, if for $b \in B$, the equation $x\alpha = b$ has a solution $x \in A$.

A mapping α of A onto B is *one-to-one* if the inverse relation α^{-1} is a mapping of B onto A. A mapping in array form is one-to-one if and only if each element of B occurs in the lower row exactly once. Making the lower row the upper and the upper row the lower, we obtain the inverse mapping array.

Thus of the two arrays

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 1 & 4\end{pmatrix}, \qquad \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 2 & 3\end{pmatrix}$$

the first defines a one-to-one mapping of $\{1, 2, 3, 4\}$ onto itself while the second does not.

Let α be a mapping of A into itself and let $a\alpha = b$. One then says that α *carries* the point a into b or *assigns* b to a. It $a\alpha = a$, then a is a *fixed* point of the mapping. All points of a set A are fixed points of the identity mapping ι_A.

Consider a one-to-one mapping α of A onto B. The inverse mapping α^{-1} is obviously a one-to-one mapping of B onto A. Since for any $a \in A$ and $b \in B$ we have $(a\alpha)\alpha^{-1} = a$ and $(b\alpha^{-1})\alpha = b$, it follows that

$$\alpha\alpha^{-1} = \iota_A, \qquad \alpha^{-1}\alpha = \iota_B. \tag{1}$$

In particular, if α is a one-to-one mapping of A onto A, then

$$\alpha\alpha^{-1} = \alpha^{-1}\alpha = \iota. \tag{2}$$

Equations (1) and (2) give rise to the following theorem.

Theorem 1. *A necessary and sufficient condition for a relation α defined on A and B to be a one-to-one onto-mapping is that α and its inverse α^{-1} satisfy equations* (1).

The necessity of equations (1) has already been proved. We merely have to show that they are sufficient. The truth of the relation $a\iota_A a$ and of the first equation in (1) implies $a\alpha\alpha^{-1}a$. This means that there is an element b in B for which $a\alpha b$ and $b\alpha^{-1}a$. Thus to each $a \in A$ there

is an element b in B for which $a \alpha b$. Suppose there were another element $b_1 \in B$ such that $a \alpha b_1$. The fact that $b_1 \alpha^{-1} a$ and $a \alpha b$ implies the truth of the relation $b_1 \alpha^{-1} \alpha b$. By (1), $\alpha^{-1} \alpha = \iota_B$ and so

$$b_1 \alpha^{-1} \alpha b \Leftrightarrow b_1 \iota_B b \Leftrightarrow b_1 = b.$$

Thus to each $a \in A$ there is just one $b \in B$ for which $a \alpha b$. Analogous reasoning shows that each b has a pre-image and so α is a mapping of A onto B. In a similar way, it can be shown that α^{-1} is a mapping of B onto A, as required.

Theorem 2. *The union (intersection) of two mappings of a set A into a set B is a mapping if and only if the two mappings are coincident.*

For, if α and β are mappings of A into B, then to each $a \in A$ there is exactly one pair of the form (a, x) and (a, y) in α and β, respectively, with x and $y \in B$. If x and y were not equal, this would imply that $\alpha \cup \beta$ contains two pairs (a, x) and (a, y) and $\alpha \cap \beta$ no pair with first element equal to a.

Let M be a set on which some operation f can be performed (for an exact definition of operation, see Section 2.1). A subset P of M is said to be *closed* under f if the application of f to the elements of P results in elements of P. Theorem 2 shows that all sets of mappings of A into A closed under union consist of a single element and so are of little interest. The same thing holds true for sets of mappings closed under intersection. By contrast, nonempty sets of mappings of A into A closed under composition are characterized by great diversity. They are called *semigroups of mappings* of A.

A *group of mappings* is a nonempty set of mappings of A into A that is closed under composition and the taking of inverses. Since the inverse of a mapping of A into A is a mapping if and only if it is one-to-one, a group of mappings comprises only one-to-one mappings.

1.4 Equivalence Relations. A binary relation in a set A is said to be an *equivalence relation* or simply an *equivalence* on A if for all x, y and z in A,

a) $x \alpha x$ (reflexivity),
b) $x \alpha y \Rightarrow y \alpha x$ (symmetry),
c) $x \alpha y \,\&\, y \alpha z \Rightarrow x \alpha z$ (transitivity),

wherein $\Rightarrow$ stands for implication.

Using the above operations on relations, we can clearly express properties a), b) and c) in the following form:

Reflexivity: $\iota \subseteq \alpha$,
Symmetry: $\alpha^{-1} \subseteq \alpha$,
Transitivity: $\alpha^2 \subseteq \alpha$,

ι being the identity relation. It is easy to see that these three conditions are equivalent to the following:

$$\iota \subseteq \alpha, \quad \alpha^{-1} = \alpha, \quad \alpha^2 = \alpha.$$

A family S of nonempty subsets of a set A is said to be a *partition* or *stratification* of A if each element of A belongs to exactly one subset in S. The subsets in S are called the *classes* or *strata* of the partition.

With each partition S we can associate a binary relation σ on A by defining $x\sigma y$ if and only if x and y belong to the same class in S. Set A

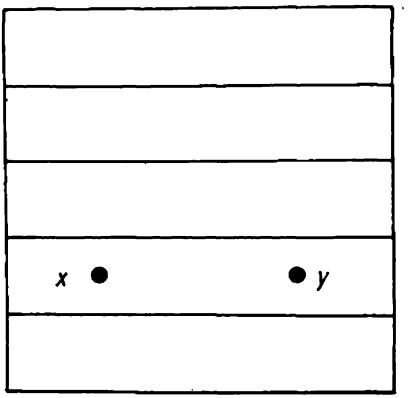

Fig. 1

has been represented in Fig. 1 by a square. The classes are the rectangles into which the square has been divided. The relation $x\sigma y$ is true if and only if x and y belong to the same rectangle. The relation σ is obviously an equivalence. It is said to be the equivalence relation *induced* by the partition S.

Let us show that each equivalence relation σ on A induces a partition S of A. To each $a \in A$, we denote by $[a]$ the set of all x for which $x\sigma a$. The subsets $[a]$ are called the *equivalence classes* of A with respect to σ or σ-equivalence classes. The fact that σ is reflexive implies that $a \in [a]$. If $b \in [a]$, then $[b] = [a]$. For, if $x \in [a]$, then $x\sigma a$. Since $b \in [a]$, $b\sigma a$ and so $a\sigma b$ (by the symmetry of σ). From $x\sigma a$ and $a\sigma b$, it follows that $x\sigma b$ (transitivity of σ) and so $[a] \subseteq [b]$. Conversely, let $x \in [b]$. Then $x\sigma b$ and $b\sigma a$ imply that $x\sigma a$ or $[b] \subseteq [a]$. Hence the family of all σ-equivalence classes of A is a partition of A. Since elements of A are equivalent if and only if they belong to the same equivalence class, the prescribed equivalence relation σ and the partition of A into disjoint classes under σ are in correspondence with one another.

The collection of all equivalence classes of set A with respect to equivalence relation σ is denoted by A/σ and is called the *factor* or *quotient set of A by σ* (or quotient set modulo σ).

The singlevalued mapping $A \to A/\sigma$ under which each element $a \in A$ is carried into the class $[a]$ containing it is called the *natural* or *canonical mapping* of A onto A/σ.

The assertion $a \sigma b$ is sometimes written in the form $a \equiv b(\sigma)$ and one says that a is congruent to b modulo σ. By the above, the equivalence $a \equiv b(\sigma)$ in set A is equivalent to $[a] = [b]$ in the quotient set A/σ.

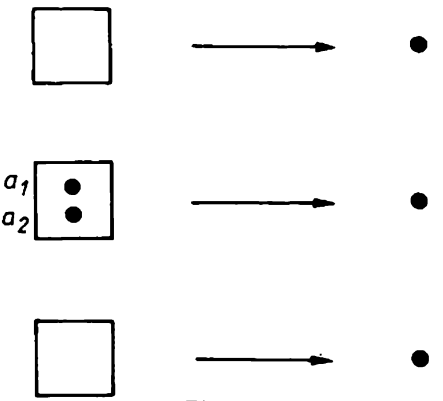

Fig. 2

We note further that the natural mapping $A \to A/\sigma$ is one-to-one if and only if the equivalence relation σ is the identity relation ι. Each ι-equivalence class of A consists of just one element. We can thus identify A with the quotient set A/ι by identifying each element of A with the set comprising that element.

Consider a mapping α of A onto B. Define a binary relation σ in A by letting $a_1 \sigma a_2$ $(a_1, a_2 \in A)$ if and only if $a_1{}^\alpha = a_2{}^\alpha$.

The relation σ is clearly an equivalence on A. The σ-equivalence classes of A are simply the complete pre-images of the elements of B. If we associate with each element of B its complete pre-image in A, we wind up with a one-to-one mapping of B onto A/σ. It is called the natural mapping and the equivalence σ is termed the *kernel equivalence* of α. In Fig. 2, the set A corresponds to the points of the squares and B to the points in the right-hand column. All the points of one square are mapped into the same point on the right. The squares are the equivalence classes of the kernel equivalence.

In conclusion we ask: Under which operations discussed above is the family of all equivalence relations on a set closed? The family of all

equivalence relations is obviously closed under the taking of inverses since every equivalence relation satisfies $\alpha^{-1} = \alpha$. Equally obvious is the following theorem.

Theorem 1. *The intersection of a family of equivalence relations (whether finite or not) on a set A is an equivalence on A.*

Let α and β be two given equivalence relations. By assumption $\iota \subseteq \alpha$ and $\iota \subseteq \beta$. Hence, $\iota \subseteq \alpha \cap \beta$. Similarly,

$$(\alpha \cap \beta)^{-1} = \alpha^{-1} \cap \beta^{-1} = \alpha \cap \beta.$$

Thus the relation $\alpha \cap \beta$ is reflexive and symmetric. Is it transitive? Let $a(\alpha \cap \beta)b$ and $b(\alpha \cap \beta)c$. This means that the relations

$$a\alpha b, \quad a\beta b, \quad b\alpha c, \quad b\beta c$$

hold true. The first and third imply $a\alpha c$ and the second and fourth $a\beta c$. Thus, $a(\alpha \cap \beta)c$, as required.

The complement of an equivalence relation does not contain ι and so is not an equivalence relation.

Theorem 2. *The composite of two equivalence relations α and β is an equivalence relation if and only if the operation $\alpha\beta$ is commutative.*

Indeed, if $\alpha\beta$ is an equivalence, then $(\alpha\beta)^{-1} = \alpha\beta$ and hence

$$\alpha\beta = \beta^{-1}\alpha^{-1} = \beta\alpha.$$

Conversely, suppose $\alpha\beta = \beta\alpha$. Since

$$x\alpha x \,\&\, x\beta x \Rightarrow x\alpha\beta x,$$

the relation $\alpha\beta$ is reflexive. From

$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} = \beta\alpha = \alpha\beta$$

it follows that $\alpha\beta$ is symmetric. Finally, the equations

$$(\alpha\beta)^2 = \alpha\beta\alpha\beta = \alpha\alpha\beta\beta = \alpha\beta$$

show that $\alpha\beta$ is transitive.

Theorem 3. *The union of equivalence relations α and β is an equivalence relation if and only if the intersection of any α-equivalence class and β-equivalence class either coincides with one of these classes or else is empty. If $\alpha \cap \beta$ is an equivalence relation, then $\alpha \cap \beta = \alpha\beta$.*

For, suppose there exists an α-equivalence class and β-equivalence class having an element a in common and that neither of these classes is a subset of the other. Then one can find elements b and c such that $b\alpha a$, $c\beta a$, $b\beta' a$ and $c\alpha' a$. The pairs (b, a) and (a, c) belong to $\alpha \cup \beta$. If $\alpha \cup \beta$ were an equivalence relation, then the pair (b, c) would also belong to $\alpha \cup \beta$. But this is impossible since $b\alpha c \,\&\, b\alpha a \Rightarrow c\alpha a$ or $b\beta c \,\&\, c\beta a \Rightarrow b\beta a$. The remaining assertions are proved in an analogous way.

Theorem 4. *The composite $\alpha\beta$ of two equivalences α and β contains α and β and is contained in any equivalence γ containing α and β.*

To see this, observe that

$$a\alpha b \Rightarrow (a\alpha b \,\&\, b\beta b) \Rightarrow a\alpha\beta b,$$

$$a\beta b \Rightarrow (a\alpha a \,\&\, a\beta b) \Rightarrow a\alpha\beta b,$$

$$a\alpha\beta b \Rightarrow \exists x(a\alpha x \,\&\, x\beta b) \Rightarrow \exists x(a\gamma x \,\&\, x\gamma b) \Rightarrow a\gamma b,$$

as required.

Corollary. *If the composition of equivalences α and β is commutative, then $\alpha\beta$ is the smallest equivalence containing α and β.*

Theorems 1—4 show that none of the operations in a relation algebra enable one to find from given equivalence relations α and β an equivalence relation containing them. Therefore of interest is one further operation namely, equivalence closure, which to a certain extent fills this gap.

Let α be a binary relation defined on a set A. The set $\sigma = A \times A$ is the "largest" relation defined on A. It is simultaneously the largest equivalence relation on A containing the given relation α. Let α^e denote the intersection of all equivalence relations defined on A that contain α. By Theorem 1, α^e is an equivalence relation. It is the smallest of equivalence relations containing α and is called the *equivalence closure* of α. If α is an equivalence relation, $\alpha^e = \alpha$. From the definition of equivalence closure, it follows that every relation α satisfies

$$(\alpha^{-1})^e = \alpha^e, \quad (\alpha^2)^e \subseteq \alpha^e.$$

Now let S be a family of relations. The equivalence closure of S is defined to be the equivalence closure of the union of S. In particular, the equivalence closure of a pair of relations α and β, which we shall temporarily denote by $\alpha \textcircled{e} \beta$, is

$$\alpha \textcircled{e} \beta = (\alpha \cup \beta)^e.$$

Equivalence closure may be characterized in the following way, which in a certain sense is more constructive.

Let S be a family of relations defined on a set A. Introduce a new relation e_A on A by setting $a e_A b$ in the following instances:

a) $a = b$;

b) $a \alpha b$ or $b \alpha a$ for some $\alpha \in S$;

c) A contains a finite collection of elements $a_1, \ldots, a_n$ $(n = 1, 2, \ldots)$ related by the conditions*)

$$a \sigma_1 a_1,\ a_1 \sigma_2 a_2,\ \ldots,\ a_n \sigma_{n+1} b, \tag{1}$$

wherein σ_i or σ_i^{-1} belong to S $(i = 1, \ldots, n + 1)$.

The relation e_A is clearly the desired equivalence closure of S. For, from a) it follows that relation e_A is reflexive. From b) it follows that e_A contains each relation $\alpha \in S$. From b) and c) it follows that e_A is symmetric and transitive. Thus e_A is an equivalence relation containing each relation in S and so containing the equivalence closure of S. On the other hand, if σ is the equivalence closure of S, then from $a e_A b$ it follows in each of the cases *a*), b) and c) that $a \sigma b$. In other words, $e_A \subseteq \sigma$ and hence $e_A = \sigma$.

The above constructive definition of equivalence closure immediately leads to the following.

Corollary 1. *The equivalence closure* $\alpha_1 \textcircled{e} \alpha_2$ *of two equivalence relations* α_1 *and* α_2 *on set* A *is true for* a *and* b *in* A *if and only if there exists a finite sequence of elements* $a_1, a_2, \ldots, a_{2n+1}$ *connected by the chain*

$$a \alpha_1 a_1 \alpha_2 a_2 \alpha_1 a_3 \alpha_2 \ldots \alpha_1 a_{2n+1} \alpha_2 b. \tag{2}$$

To prove this, we merely have to show that each of the conditions a), b) and c) leads to a suitable condition such as (2). Condition a) clearly implies the condition $a \alpha_1 a \alpha_2 a$, which is of form (2) and condition b) implies $a \alpha_1 a \alpha_2 b$ or $a \alpha_1 b \alpha_2 b$. Since the relations α_1 and α_2 are symmetric, the words "or σ_i^{-1}" may be discarded. Further, if the segment $a_k \alpha_i a_{k+1} \alpha_i a_{k+2}$ occurs in the chain (1), then by virtue of the transitivity of α_i $(i = 1, 2)$ the segment may be replaced by $a_k \alpha_i a_{k+2}$, etc.

A reflexive symmetric relation is called a *pseudo-equivalence*. To construct the equivalence closure of a family of pseudo-equivalence relations is somewhat simpler compared to the general case. We state these simplifications in the form of a special corollary.

---

*) The relations (1) will be written in the form $a \sigma_1 a_1 \sigma_2 a_2 \ldots a_n \sigma_{n+1} b$ [*Trans.*].

Corollary 2. *The equivalence closure of a family S of pseudo-equivalences on a set A is true for elements a and b in A if and only if there exists a finite sequence of elements $a_1, a_2, \ldots, a_n$ in A connected by the chain*

$$a\sigma_1 a_1 \sigma_2 a_2 \sigma_3 \ldots a_n \sigma_{n+1} b,$$

where $\sigma_1, \ldots, \sigma_{n+1}$ are any relations in S (possibly repeated).
The proof is exactly the same as that of Corollary 1.

The construction of the equivalence closure can be presented in a different form by making use of the following remarks.

Remark 1. The composite $\sigma_1\sigma_2\ldots\sigma_k$ of reflexive relations $\sigma_1, \ldots, \sigma_k$ is reflexive and contains each of the enumerated relations.

In view of the fact that composition of relations is associative, it suffices to prove the remark for just two factors. We have

$$x\sigma_1 y \Rightarrow (x\sigma_1 y \,\&\, y\sigma_2 y) \Rightarrow x(\sigma_1\sigma_2)y,$$

and so $\sigma_1 \subseteqq \sigma_1\sigma_2$. Similarly, from

$$x\sigma_2 y \Rightarrow (x\sigma_1 x \,\&\, x\sigma_2 y) \Rightarrow x(\sigma_1\sigma_2)y$$

we find that $\sigma_2 \subseteqq \sigma_1\sigma_2$. Finally, for arbitrary x,

$$x\sigma_1 x \,\&\, x\sigma_2 x \Rightarrow x(\sigma_1\sigma_2)x,$$

as required.

Remark 2. If equivalence relation σ contains relations $\sigma_1, \sigma_2, \ldots, \sigma_k$, then it contains their composition.

Again it suffices to consider the case of two factors. We have

$$x(\sigma_1\sigma_2)y \Rightarrow \exists z(x\sigma_1 z \,\&\, z\sigma_2 y) \Rightarrow \exists z(x\sigma z \,\&\, z\sigma y) \Rightarrow x\sigma\sigma y \Rightarrow x\sigma y,$$

as required.

Theorem 5. *The equivalence closure of a family of equivalence relations σ_α coincides with the union σ of all possible compositions of these equivalence relations.*

Indeed, by Remark 1, σ contains all of the equivalences σ_α and by Remark 2, σ is contained in all equivalences containing σ_α.

1.5 Partial and Linear Orderings. Besides equivalence relations, another important kind of binary relation is that of partial ordering. A binary relation ϱ in a set A is said to be a *partial ordering* if it is reflexive, transitive (see Section 1.4) and antisymmetric:

$$x\varrho y \,\&\, y\varrho x \Rightarrow x = y$$

for any elements x and y in A. Thus, a binary relation in A is a partial ordering if and only if it has the following three properties:

$$\iota \subseteq \varrho, \quad \varrho \cap \varrho^{-1} \subseteq \iota, \quad \varrho^2 \subseteq \varrho.$$

It is easy to verify that this set of conditions is equivalent to the following:

$$\iota \subseteq \varrho, \quad \varrho \cap \varrho^{-1} = \iota, \quad \varrho^2 = \varrho.$$

Observe that for each partial ordering ϱ of a set A, the inverse relation ϱ^{-1} will also be a partial ordering. It is usually called the *dual* to ϱ.

A partial ordering of a set A is frequently denoted by the symbol $\leqq$. If $a \leqq b$ for elements a and b in A, then one says that a is less than or equal to b, or that a is contained in or is equal to b. If $a \leqq b$ and $a \neq b$, one writes $a < b$ and says: a is strictly less than b or a is contained strictly in b.

For instance, let $\mathfrak{U}$ be the collection of all subsets of a fixed set U. The inclusion relation $x \subseteq y$ for elements x and y in $\mathfrak{U}$ (see Section 1.1) is obviously a partial ordering of $\mathfrak{U}$.

Elements a and b of a set A are said to be *comparable* under a partial ordering $\leqq$ defined on A if $a \leqq b$ or $b \leqq a$. A partial ordering of a set A is said to be a *linear* (*total, simple*) *ordering* if every pair of elements a and b in A are comparable under $\leqq$.

Let $\leqq$ be a partial ordering of a set A and let $B \subseteq A$, An element $a \in A$ is an *upper bound* for B in A if $b \leqq a$ for all $b \in B$. An element $a \in A$ is said to be *greatest* in A if a is an upper bound for A itself. An element $m \in A$ is said to be *maximal* in A if each element x in A is not comparable with m or $x \leqq m$.

If a set A has a greatest element a, then a will obviously be the only maximal element in A. In similar fashion, one can define the notions of lower bound of a subset B in A and least and minimal elements of A.

For example, if $A = \{1, 2, 3, 4, 5\}$ and $x \leqq y$ means that y is divisible by x without remainder, then 1 is the least element of A and 3, 4 and 5 are maximal elements of A. In this example, A has no greatest element.

A set A on which there is defined a partial (linear) ordering is said to be *partially* (*linearly*) *ordered*. A linearly ordered set is also called a *chain*.

If $\langle A, \leqq \rangle$ is a partially ordered set and B is a subset of A, then the relation

$$x \varrho y \Leftrightarrow x \leqq y \quad (x, y \in B)$$

will clearly be a partial ordering of B. One usually denotes it by the same symbol used for the given partial ordering in A. Thus every subset B of a partially ordered set $\langle A, \leqq \rangle$ is partially ordered under the relation $\leqq$.

Zorn's Lemma. *Each element of a nonempty partially ordered set* $\langle A, \leqq \rangle$ *in which each chain* $\langle B, \leqq \rangle$ $(B \subseteq A)$ *has an upper bound is contained in some maximal element.*

This assertion is logically equivalent to the Axiom of Choice (see Section 2.5) and will therefore be taken to be an axiom.

A linearly ordered set A is said to be *well-ordered* if every nonempty subset B of A has a least element.

Every element a of a well-ordered set A which is not greatest in A has a unique element a' such that $a < a'$ and $a' \leqq x$ for every $x \in A$ greater than a. The element a' is called the *direct successor* of a. Element a in turn is called the *direct predecessor* of a'. An element of A having no direct predecessor in A is said to be a *boundary element.*

Zorn's Lemma is also logically equivalent to the following axiom.

Well-Ordering Principle.

Every nonempty set can be well-ordered. Statements concerning elements of a well-ordered set A can often be proved by the *method of transfinite induction* which amounts to the following. Suppose x_0 is the least element of a set A and $P(x)$ is some statement about element $x \in A$. If $P(x_0)$ is true and if the truth of $P(x)$ for all elements $x \in A$ strictly less than a implies the truth of statement $P(a)$, then it can be asserted that $P(x)$ is true for all $x \in A$. To see this, let B be the set of elements y in A for which statement $P(y)$ is false. If $B \neq \varnothing$, then B has a least element which we denote by a. Since $P(x_0)$ is true, $a \neq x_0$ and therefore $a > x_0$. For all elements x strictly less than a, statement $P(x)$ is true. By hypothesis, $P(a)$ must also be true and so we have arrived at a contradiction.

1.6 Multivalued and Partial Mappings. The concepts of mapping and equivalence relation considered in the preceding subsections have useful generalizations which we shall now present.

Let α be an arbitrary relation from set A to set B. As in the case of mapping, an element $b \in B$ is the *image* of $a \in A$ if $a \alpha b$ holds true. In contrast to our "real" mappings, there may now in general be elements in A having no image in B at all or having several images in B. For that reason, such a relation α is called a *partial multimapping* of A into B.

If α is a partial multimapping of A into B, then as in the case of ordinary mappings, one defines

$$a\alpha = \{x \mid a\alpha x\} \quad (a \in A),$$

$$A_1\alpha = \{x \mid a\alpha x,\ a \in A_1\} \quad (A_1 \subseteq A).$$

Thus $A_1\alpha$ is the set of all images of every element in A_1. The set $B\alpha^{-1}$ is called the *domain* of the partial multimapping α and $A\alpha$ its *range*.

If a partial multimapping α from set A into set B is such that for each element $a \in A$ the set $a\alpha$ is either empty or has just a single element, then α is called *a partial mapping* of A into B.

Each pair of relations α and β defined on sets A and B and B and C, respectively, clearly satisfies

$$(A_1\alpha)\beta = A_1(\alpha\beta) \quad (A_1 \subseteq A).$$

Therefore instead of $(A_1\alpha)\beta$ and $A_1(\alpha\beta)$, one usually writes simply $A_1\alpha\beta$.

As before, the symbols ι_A and ι_B stand for the identity relations in A and B, respectively. Let α be a relation from A to B. For natural reasons, one says that α is a partial multimapping of A into B *defined on* A if $\iota_A \subseteqq \alpha\alpha^{-1}$. On the other hand, if $\alpha\alpha^{-1} \subseteqq \iota_A$, then α is a *strict* partial multimapping of A into B. Finally, if $\alpha^{-1}\alpha \subseteqq \iota_B$, then α is a *singlevalued* partial multimapping of A into B.

In particular, a relation α is an ordinary mapping of A into B if α is defined on A and is singlevalued. A partial mapping of A into B is a singlevalued partial multimapping of A into B.

Sets of the form $b\alpha^{-1}$ $(b \in B)$ are termed the *α-classes of A* and those of the form $a\alpha (a \in A)$ the *α^{-1}-classes of B*. The *quotient sets* A/α and B/α^{-1} are the sets having as respective elements the α-classes of A and α^{-1}-classes of B.

It is easy to see that the union of all α-classes of A is the domain of the partial multimapping α. It need not coincide with A. The classes in A/α may also overlap partially and so their totality does not necessarily form a partition of the the domain of α. Consider for example, the relation α defined on the sets

$$A = \{0, 1, 2, 3\}, \quad B = \{a, b, c\}$$

consisting of the pairs $(0, a)$, $(0, b)$, $(1, a)$ and $(2, b)$. The domain of α is

$$B\alpha^{-1} = \{0, 1, 2\},$$

and the quotient set A/α consists of the classes

$$a\alpha^{-1} = \{0, 1\}, \quad b\alpha^{-1} = \{0, 2\},$$

obviously having the element 0 in common.

Let α be a relation from set A to set B. If for any a, a_1 in A and b, b_1 in B, it follows from $a\alpha b_1$, $a_1\alpha b_1$ and $a_1\alpha b$, that $a\alpha b$, i.e., if

$$\alpha\alpha^{-1}\alpha \subseteq \alpha.$$

then α is said to be *difunctional.* Since we always have $\alpha \subseteq \alpha\alpha^{-1}\alpha$, a relation α is difunctional if and only if $\alpha\alpha^{-1}\alpha = \alpha$.

Theorem 1 (Riguet [58]). *Let α be a relation from set A to set B. Then the α-classes of A or the α^{-1}-classes of B are pairwise disjoint if and only if α is a difunctional relation.*

Proof. Suppose the α-classes of A are pairwise disjoint and that $a\alpha b_1$, $a_1\alpha b_1$ and $a_1\alpha b$ (a and a_1 in A; b and b_1 in B). Since $a_1 \in b\alpha^{-1}$ and $a_1 \in b_1\alpha^{-1}$, we have $b\alpha^{-1} = b_1\alpha^{-1}$. But $a \in b_1\alpha^{-1}$. Therefore $a \in b\alpha^{-1}$ or $a\alpha b$ so that α is difunctional. The case where the α^{-1}-classes of B are disjoint is handled in a similar way since $\alpha\alpha^{-1}\alpha \subseteq \alpha$ and $\alpha^{-1}\alpha\alpha^{-1} \subseteq \alpha^{-1}$ are equivalent.

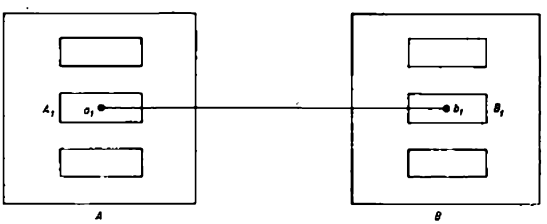

Fig. 1

Conversely, suppose α is a difunctional relation and that for some a and a_1 in A, $a\alpha \cap a_1\alpha \neq \varnothing$. In other words, for suitable b_1 in B, we have $a\alpha b_1$ and $a_1\alpha b_1$. Let $b \in a_1\alpha$. Using the fact that α is difunctional, we find that $b \in a\alpha$, i.e., $a_1\alpha \subseteq a\alpha$. In a similar way, we can establish the inclusion $a\alpha \subseteq a_1\alpha$. The theorem is proved.

Consider now an arbitrary difunctional relation α defined on the pair of sets A and B. A class $B_1 \in B/\alpha^{-1}$ is said to be the *image* of class $A_1 \in A/\alpha$ (see Fig. 1) if one can find elements $a_1 \in A_1$ and $b_1 \in B$ such that $a_1\alpha b_1$. By Theorem 1, the fact that $a_1\alpha b_1$ implies that $a\alpha b$ for any $a \in A_1$ and $b \in B_1$. Therefore the mapping of A/α into B/α^{-1} just established does not depend on the choice of representatives a_1 and b_1 and is a one-to-one mapping of A/α onto B/α^{-1}. It is termed the *natural* or *canonical mapping* of A/α onto B/α^{-1} induced by the relation α.

Conversely, suppose S is an arbitrary family of pairwise disjoint nonempty subsets of A and T a family of pairwise disjoint nonempty subsets of B. In addition, suppose that there exists a one-to-one mapping σ of the family S onto T. Let α denote the totality of pairs (a, b) such

that $a \in A_1 \in S$ and $b \in A_1\sigma \in T$. It is easy to prove that the relation α so defined is difunctional, $A/\alpha = S$, $B/\alpha^{-1} = T$ and that σ is the natural mapping of A/α onto B/α^{-1} induced by α.

The above discussion shows that difunctional relations are those relations induced by one-to-one correspondences existing between families of pairwise disjoint subsets of prescribed sets A and B.

1.7 Cardinal and Ordinal Numbers. With each set A we associate a number $|A|$, called the *cardinality* of the set, so that $|A| = |B|$ if and only if there exists a one-to-one mapping of A *onto* B. In particular, the empty set $\varnothing$ is assigned cardinality 0 and the set $\{a_1, \ldots, a_n\}$ consisting of n elements ($n = 1, 2, \ldots$) the number n.

Terms synonymous with cardinality are *cardinal number* and *power*. The cardinal of the set of all natural numbers is usually denoted by $\aleph_0$ (aleph-null) and that of the set of all real numbers by $\aleph$ (aleph). $\aleph$ is also said to be the *cardinality of the continuum*. Sets having cardinality $\aleph_0$ are called *countable*. The cardinals corresponding to the natural numbers are called *finite*. All others are called *transfinite cardinals*.

Let $\mathfrak{a} = |A|$ and $\mathfrak{b} = |B|$. We define $\mathfrak{a} \leqq \mathfrak{b}$ if there exists a one-to-one mapping of A *into* B. It is easy to show that the definition is independent of the choice of A and B and so expresses a relation between cardinal numbers.

Theorem 1. *The relation $\leqq$ is a linear ordering in any set of cardinal numbers.*

From the definition of $\leqq$, it follows that this relation is reflexive and transitive. Its antisymmetry is a consequence of the following Cantor-Bernstein theorem.

Theorem 2.*) *If α is a one-to-one mapping of set A onto the subset $\alpha(A)$ of A, then to every set $C \subseteq A - \alpha(A)$, there exists a one-to-one mapping α^* of A onto the set $\alpha^*(A) = C \cup \alpha(A)$.*

Indeed, let $\alpha^0(C) = C$ and $\alpha^{n+1}(C) = \alpha(\alpha_n(C))$ and consider the set

$$S = \bigcup_{n=0}^{\infty} \alpha_n(C).$$

The mapping

$$\alpha^*(x) = \begin{cases} x \text{ for } x \in S, \\ \alpha(x) \text{ for } x \in (A - S) \end{cases}$$

*) The statement of the Cantor-Bernstein theorem in this form and its proof are due to Reichbach (see M. Reichbach, Colloq. Math., 3, No. 2 (1955) p. 163).

is the identity on S and coincides with α on the set $A - S$. Since $S = C \cup \alpha(S)$, we have

$$S \cap \alpha(A - S) = \big(C \cap \alpha(A - S)\big) \cup \big(\alpha(S) \cap \alpha(A - S)\big) = \varnothing .$$

Therefore α^* is a one-to-one mapping of A onto the set

$$\alpha^*(A) = S \cup \alpha(A - S) = C \cup \alpha(S) \cup \alpha(A - S) = C \cup \alpha(A).$$

Suppose now that $\mathfrak{a} \leqq \mathfrak{b}$ and $\mathfrak{b} \leqq \mathfrak{a}$. This means that there exist one-to-one mappings

$$\alpha: A \to \alpha(A), \quad \alpha(A) \subseteq B, \qquad \beta: B \to \beta(B), \quad \beta(B) \subseteq A .$$

The composition $\gamma = \alpha\beta$ (see Section 1.3) is a one-to-one mapping of A onto $\gamma(A) = \beta\big(\alpha(A)\big)$. By Theorem 2, there exists a one-to-one mapping γ^* of A onto the set $C \cup \gamma(A)$ for any $C \subseteq \big(A - \gamma(A)\big)$. Choosing $C = \beta(B) - \gamma(A)$, we have $\gamma^*(A) = \beta(B)$. Hence $\gamma^*\beta^{-1}$ is a one-to-one mapping of A onto B and therefore $\mathfrak{a} = \mathfrak{b}$.

Thus $\leqq$ is a reflexive antisymmetric transitive relation for cardinal numbers. A little later on we shall show that any two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ are comparable. And therefore the relation $\leqq$ is a linear ordering in any set of cardinal numbers.

A linearly ordered set A is said to be *isomorphic* to a linearly ordered set B if there exists a one-to-one mapping φ of A *onto* B preserving the linear ordering. That is, $a \leqq b$ in A implies $a\varphi \leqq b\varphi$. A mapping φ having these properties is called an *isomorphism of A onto B.*

With each linearly ordered set A we associate an entity $o(A)$ called its *order type* so that $o(A) = o(B)$ if and only if the linearly ordered sets A and B are isomorphic.

Since the equality $o(A) = o(B)$ of order types implies the equality of cardinals $|A| = |B|$, to each order type there corresponds a cardinal number said to be the cardinal of that type.

Each set $\{a_1, \ldots, a_n\}$ consisting of n elements $(n = 1, 2, \ldots)$ admits $n!$ permutations and therefore can be linearly ordered in $n!$ different ways. But all of the resultant linearly ordered sets have the exact same order type. It is therefore denoted by n. The empty set is assigned order type 0. The set of natural numbers $N = \{1, 2, 3, \ldots\}$ with its natural order $1 < 2 < 3 \cdots$ is assigned the order type ω, while the set $\langle N, \leqq^{-1}\rangle$ with the dual linear order $\cdots < 3 < 2 < 1$ has the order type ω^*.

The order type of a well-ordered set A is called an *ordinal number* or simply an *ordinal.* The ordinals which are not natural numbers are called *transfinite ordinals.*

Hence it follows that the types n $(n = 0, 1, 2, \ldots)$ and ω are ordinals but the order type ω^* is not.

If $\langle A, \leqq \rangle$ is any well-ordered set and a is an element of A, then the well-ordered set

$$P_a = \langle \{x \mid x \in A,\ x < a\}, \leqq \rangle$$

of elements of A strictly less than a is called the *initial segment* determined by a.

Lemma. *If φ is an isomorphism of a well-ordered set A onto a subset B of A, then $\varphi(a) \geqq a$ for each element $a \in A$.*

For, suppose that $\varphi(a) < a$ for some $a \in A$. We may assume that a is the least element in A with this property. Setting $b = \varphi(a)$, we have $\varphi(b) < \varphi(a)$ since $b < a$. Thus $\varphi(b) < b$, which contradicts the choice of element a.

Corollary. *A well-ordered set cannot be isomorphic to any initial segment of it.*

Let $\alpha = o(A)$ and $\beta = o(B)$ be given ordinals. Define $\alpha < \beta$ if the set A is isomorphic to some initial segment of set B. Taking the union of the relations $<$ and $=$ (see Section 1.2), we obtain a new relation which we denote by $\leqq$.

Let $W(\alpha)$ be the set of all ordinals strictly less than the ordinal α. For example, $W(0) = \varnothing$, $W(1) = \{0\}$, $W(n) = \{0, 1, \ldots, n-1\}$ and $W(\omega) = \{0, 1, 2, \ldots\}$.

Theorem 3. *The set $\langle W(\alpha), \leqq \rangle$ is well-ordered and has order type α.*

Proof. The relation $\leqq$ defined above is reflexive and transitive. It is also antisymmetric since if $\beta \leqq \gamma$ and $\gamma \leqq \beta$, $\gamma \neq \beta$, we must have $\beta < \beta$ by the transitivity. This contradicts the corollary to the lemma. Let us show that any two unequal ordinals β and γ in $W(\alpha)$ are comparable.

Let $\alpha = o(A)$, $\beta = o(B)$ and $\gamma = o(C)$. Since $\beta < \alpha$ and $\gamma < \alpha$, each of the well-ordered sets B and C is isomorphic to an initial segment of A. Denote these segments by P_b and P_c, respectively. The elements b and c of A corresponding to these segments are comparable. If, for example, $b < c$, then $\beta < \gamma$. Thus, β and γ are also comparable.

For each element b in A, the ordinal $\beta = o(P_b)$ belongs to $W(\alpha)$ and so the correspondence $b \to \beta$ is clearly an isomorphism of A onto $W(\alpha)$. Therefore $W(\alpha)$ is a well-ordered set and $\alpha = o(W(\alpha))$. Theorem 3 is proved.

Thus a well-ordered set A of type α is isomorphic to the set of ordinals $W(\alpha)$. The elements of A may therefore be indexed by means of the numbers in $W(\alpha)$; thus $A = \{a_\xi \mid \xi < \alpha\}$, the subscript ξ being the order type of the segment P_{a_ξ} of A corresponding to element a_ξ.

Theorem 4. *Two ordinals α and β are always comparable.*

Proof. Let $A = W(a)$, $B = W(\beta)$, $D = A \cap B$ and $\delta = o(D)$. Since D is a well-ordered set, δ is an ordinal number. Let us show that $\delta \leqq \alpha$.

If $D = A$, then $\delta = \alpha$. Suppose $D \subset A$. It is easy to show that D is an initial segment of A and so $\delta < \alpha$. Indeed, if $\xi \in D$ and $\eta \in A - D$, then either $\xi < \eta$ or $\eta < \xi$. The case $\eta < \xi$ is impossible since then we would have $\eta < \xi < \alpha$ and $\eta < \xi < \beta$ which would mean the number η belongs to D. Hence, $\xi < \eta$. Having chosen η to be least in $A - D$, we find that $D = W(\eta)$, i.e., D is an initial segment. Since $\eta = o(W(\eta)) = o(D) = \delta$, it follows that $D = W(\delta)$.

Thus, $\delta \leqq \alpha$ and $\delta \leqq \beta$. The combination $\delta < \alpha$ and $\delta < \beta$ is impossible since $\delta \notin D = W(\delta)$. Therefore either $\delta = \alpha$ and $\delta = \beta$ in which case $\alpha = \beta$, or $\delta = \alpha$ and $\delta < \beta$ and $\alpha < \beta$, or else $\delta < \alpha$ and $\delta = \beta$ and $\beta < \alpha$. Theorem 4 is proved.

From the Well-Ordering Principle and Theorem 4, we can conclude that every pair of cardinals $\mathfrak{a}$ and $\mathfrak{b}$ are also comparable. For, according to the principle $\mathfrak{a}$ and $\mathfrak{b}$ are the cardinals of some well-ordered sets A and B of types α and β, respectively. If $\alpha = \beta$, then $\mathfrak{a} = \mathfrak{b}$. But if $\alpha < \beta$, then A is isomorphic to an initial segment of B and therefore $\mathfrak{a} \leqq \mathfrak{b}$. Similarly, $\mathfrak{a} \geqq \mathfrak{b}$ for $\alpha > \beta$. Theorem 1 has been completely proved.

A cardinal number $\mathfrak{a}$ is said to be finite if $\mathfrak{a} < \aleph_0$ and infinite (or an aleph) if $\mathfrak{a} \geqq \aleph_0$. An ordinal number is said to be finite or infinite depending on whether its cardinality is finite or infinite.

Theorem 5. *Each nonempty set W of ordinal numbers has a least element.*

In other words, every set of ordinals is well-ordered. To see this, let $\alpha \in W$. If α is not least in W, take the set $W \cap W(\alpha)$. Being a subset of $W(\alpha)$, it is well-ordered and so possesses a least element α_0. The number α_0 will be the least element of W. For, if $\beta \in W$, then $\alpha_0 \leqq \beta$ when $\alpha \leqq \beta$ while for $\beta < \alpha$, $\beta \in W \cap W(\alpha)$ and so $\alpha_0 \leqq \beta$ also.

Corollary. *Every set A of cardinals is also well-ordered.*

Indeed, with each cardinal number $\mathfrak{a}$ in the given set A, we can associate some ordinal number α having cardinality $\mathfrak{a}$. This will define an isomorphism of A onto the set of ordinal numbers.

In concluding this subsection, we shall prove Cantor's theorem.

Theorem 6. *The set $\mathscr{P}(A)$ of all subsets of a set A has cardinality strictly greater than the cardinality of A.*

To this end, associate with each element $a \in A$ the singleton subset $\{a\}$ of A. This defines a one-to-one mapping of A into $\mathscr{P}(A)$. Therefore $|A| \leqq |\mathscr{P}(A)|$.

Suppose there existed a one-to-one mapping φ of A onto $\mathscr{P}(A)$. Let

$$M = \{a \in A \mid a \notin \varphi(a)\}.$$

M is a subset of A and therefore $M \in \mathscr{P}(A)$. Hence, there must be an element $m \in A$ such that $\varphi(m) = M$. We now obtain a contradiction. If $m \in M$, then $m \notin \varphi(m) = M$ and if $m \notin M$, then $m \in \varphi(m) = M$.

Corollary. *To each cardinal* $\mathfrak{a}$ (*ordinal* α), *there is a least cardinal* $\mathfrak{b}$ (*ordinal* β) *such that* $\mathfrak{a} < \mathfrak{b}$ ($\alpha < \beta$).

For, by Theorem 6, there exists a cardinal number $\mathfrak{b}$ with $\mathfrak{a} < \mathfrak{b}$. By the corollary to Theorem 5, there is a least cardinal number among those strictly greater than $\mathfrak{a}$ and less than or equal to $\mathfrak{b}$. The assertion for ordinal numbers is proved in a similar way.

Operations on cardinal and ordinal numbers will be considered in detail in Section 2.6. At this point, we shall merely define what is meant by the sum of two ordinal numbers.

Let $\alpha = o(A)$ and $\beta = o(B)$ be two arbitrary order types with $A \cap B = \varnothing$. The sum $\alpha + \beta$ is defined as the order type of the union of sets A and B linearly ordered as follows: if $a \in A$ and $b \in B$, then $a < b$; otherwise the linear ordering in each of the sets A and B is preserved.

For example, $1 + \omega$ is the order type of the union $\{1\} \cup \{2, 3, \ldots\}$ linearly ordered according to the natural order $1 < 2 < 3 < \cdots$ Therefore $1 + \omega = \omega$. The sum $\omega + 1$ is the order type of the set $\{a_2, a_3, \ldots, a_1\}$ in which $a_2 < a_3 < \cdots < a_1$. Since this set has a greatest element, $\omega + 1 \neq \omega$. All of the types ω, $\omega + 1$, $\omega + 2$, ... are clearly distinct.

From the definition, it follows that *the sum* $\alpha + \beta$ *of two ordinal numbers* α *and* β *is an ordinal number.*

Problems and Complements

1. Let α and β be mappings of the set $A = \{1, 2, 3, 4, 5, 6\}$ into itself defined by the arrays

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 3 & 6 & 6 \end{pmatrix}.$$

Show that

$$\alpha\alpha^{-1}\alpha = \alpha, \quad \beta\beta^{-1} = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 \\ 1 & 4 & 2 & 3 & 1 & 4 & 5 & 6 & 5 & 6 \end{pmatrix},$$

$$A/\alpha^e = \{\{1,2, 3, 4\}, \{5, 6\}\}, \quad A/\beta^e = \{\{1, 3, 4\}, \{2\}, \{5, 6\}\},$$

$$A/(\alpha \cup \beta)^e = \{\{1, 2, 3, 4\}, \{5, 6\}\},$$

where α^e is the equivalence closure of α.

2. Let α be a difunctional relation defined on the pair A, A. If α is a pseudo-equivalence, then A/α is a partition of A and the induced mapping is the identity mapping of A/α onto itself. Thus, relation α is an equivalence relation if and only if it is reflexive and symmetric.

3. A relation α is an equivalence relation if and only if $\iota \subseteq \alpha$ and $\alpha\alpha^{-1} \subseteq \alpha$.

4. A linearly ordered set is said to be *unbounded* if it is neither bounded from below or above and *dense* if for every pair of its elements a and b with $a < b$, there exists an element c between them: $a < c < b$. All countable unbounded dense sets are isomorphic (Hausdorff [18], Section 11).

2. Models and Algebras

2.1 n-ary Relations and Functions. The *cartesian product* $A_1 \times A_2 \times \cdots \times A_n$ of a family of sets $A_1, A_2, \ldots, A_n$ is the collection of sequences of the form $(a_1, a_2, \ldots, a_n)$ with $a_1 \in A_1, \ldots, a_n \in A_n$. Each subset R of $A_1 \times A_2 \times \cdots \times A_n$ is said to be a *relation* defined on the family $A_1, A_2, \ldots, A_n$. A cartesian product $A_1 \times A_2 \times \cdots \times A_n$ in which $A_1 = A_2 = \cdots = A_n = A$ is called the *cartesian n-th power of A* and is denoted by A^n. A relation R defined on the family $A, A, \ldots, A$ is said to be an *n-ary relation in A*.

A partial mapping α from $A_1 \times A_2 \times \cdots \times A_n$ into B (see Section 1.6) is called a *partial function* from $A_1 \times A_2 \times \cdots \times A_n$ into B. A mapping α of $A_1 \times A_2 \times \cdots \times A_n$ into B is a *function* from $A_1 \times A_2 \times \cdots \times A_n$ into B. In conformity with Section 1.3, the element $b \in B$ corresponding to $(a_1, \ldots, a_n)$, $a_1 \in A_1, \ldots, a_n \in A_n$, under α is the *value of the function* α at the point $(a_1, \ldots, a_n)$ and is denoted by $(a_1, \ldots, a_n)\alpha$. Instead of $(a_1, \ldots, a_n)\alpha$ one often uses also the notation $\alpha(a_1, \ldots, a_n)$, $a_1 \ldots a_n \alpha$ and $a_1 \alpha a_2 \alpha \ldots \alpha a_n$. For $n = 1$ and $n = 2$, the notation αa and $a_1 \alpha a_2$ is almost always employed.

The distinction between a function and a partial function is quite important. A function α from $A_1 \times A_2$ into B has a uniquely defined value $\alpha(a_1, a_2) = a_1 \alpha a_2$ at the point (a_1, a_2) for each $a_1 \in A_1$ and $a_2 \in A_2$. Whereas if α is a partial function from $A_1 \times A_2$ into B, then for certain $a_1 \in A_1$ and $a_2 \in A_2$ there need not exist a value $a_1 \alpha a_2$. In that case, one says that $a_1 \alpha a_2$ is not determined or that the expression $a_1 \alpha a_2$ has an *indeterminate value*.

A partial function from A^n into B is called a *partial n-ary function on A* with values in B. A function from A^n into B is called an *n-ary function on A* with values in B. An n-ary function on A with values in A is called an *n-ary operation on A*.*) An *n-ary partial operation on set A* is defined in a similar way. The collection of points at which a given partial function α has a definite value is called the *domain* of α.

*) The number n is sometimes called the *arity* of the n-ary operation [*Trans.*].

2. Let α be a difunctional relation defined on the pair A, A. If α is a pseudo-equivalence, then A/α is a partition of A and the induced mapping is the identity mapping of A/α onto itself. Thus, relation α is an equivalence relation if and only if it is reflexive and symmetric.

3. A relation α is an equivalence relation if and only if $\iota \subseteq \alpha$ and $\alpha\alpha^{-1} \subseteq \alpha$.

4. A linearly ordered set is said to be *unbounded* if it is neither bounded from below or above and *dense* if for every pair of its elements a and b with $a < b$, there exists an element c between them: $a < c < b$. All countable unbounded dense sets are isomorphic (Hausdorff [18], Section 11).

2. Models and Algebras

2.1 n-ary Relations and Functions. The *cartesian product* $A_1 \times A_2 \times \cdots \times A_n$ of a family of sets $A_1, A_2, \ldots, A_n$ is the collection of sequences of the form $(a_1, a_2, \ldots, a_n)$ with $a_1 \in A_1, \ldots, a_n \in A_n$. Each subset R of $A_1 \times A_2 \times \cdots \times A_n$ is said to be a *relation* defined on the family $A_1, A_2, \ldots, A_n$. A cartesian product $A_1 \times A_2 \times \cdots \times A_n$ in which $A_1 = A_2 = \cdots = A_n = A$ is called the *cartesian n-th power of A* and is denoted by A^n. A relation R defined on the family $A, A, \ldots, A$ is said to be an *n-ary relation in A*.

A partial mapping α from $A_1 \times A_2 \times \cdots \times A_n$ into B (see Section 1.6) is called a *partial function* from $A_1 \times A_2 \times \cdots \times A_n$ into B. A mapping α of $A_1 \times A_2 \times \cdots \times A_n$ into B is a *function* from $A_1 \times A_2 \times \cdots \times A_n$ into B. In conformity with Section 1.3, the element $b \in B$ corresponding to $(a_1, \ldots, a_n)$, $a_1 \in A_1, \ldots, a_n \in A_n$, under α is the *value of the function* α at the point $(a_1, \ldots, a_n)$ and is denoted by $(a_1, \ldots, a_n)\alpha$. Instead of $(a_1, \ldots, a_n)\alpha$ one often uses also the notation $\alpha(a_1, \ldots, a_n)$, $a_1 \ldots a_n \alpha$ and $a_1 \alpha a_2 \alpha \ldots \alpha a_n$. For $n = 1$ and $n = 2$, the notation αa and $a_1 \alpha a_2$ is almost always employed.

The distinction between a function and a partial function is quite important. A function α from $A_1 \times A_2$ into B has a uniquely defined value $\alpha(a_1, a_2) = a_1 \alpha a_2$ at the point (a_1, a_2) for each $a_1 \in A_1$ and $a_2 \in A_2$. Whereas if α is a partial function from $A_1 \times A_2$ into B, then for certain $a_1 \in A_1$ and $a_2 \in A_2$ there need not exist a value $a_1 \alpha a_2$. In that case, one says that $a_1 \alpha a_2$ is not determined or that the expression $a_1 \alpha a_2$ has an *indeterminate value*.

A partial function from A^n into B is called a *partial n-ary function on A* with values in B. A function from A^n into B is called an *n-ary function on A* with values in B. An n-ary function on A with values in A is called an *n-ary operation on A*.*) An *n-ary partial operation on set A* is defined in a similar way. The collection of points at which a given partial function α has a definite value is called the *domain* of α.

*) The number n is sometimes called the *arity* of the n-ary operation [*Trans.*].

For example, the ordinary sum $a + b$ and product ab of natural numbers a and b are binary operations on the set N of natural numbers. Subtraction is not always possible within the framework of the natural numbers. Therefore subtraction in N is to be viewed as a partial binary operation. The domain of subtraction in N consists of the pairs (a, b) in which $a \geqq b$.

Subtraction in the set R of all rational numbers is always well-defined. Division is a partial binary operation both in N and in R. However the domain of the division operation in R consists of all pairs (a, b), with $b \neq 0$, whereas the domain for division in N consists of the pairs (a, b), $b \neq 0$, for which a is a multiple of b.

In these examples, the same symbol was used to denote different operations. One of them defined on N was partial while another on R was well-defined everywhere. According to the above definitions, partial functions α from P into A and β from Q into B will be termed *equal* if and only if $P = Q$, $A = B$ and for each $x \in P$ the expression $\beta(x)$ is determinate whenever $\alpha(x)$ is and the two have the same value. For example, the unary partial functions $f(x)$ and $g(x)$ from N into N defined by

$$f(x) = 0 \cdot (x - 1), \qquad g(x) = 0 \cdot (x - 2)$$

are identically zero on their domains of definition. However the domain of f consists of the naturals $x \geqq 1$ while the domain of g consists of the naturals $x \geqq 2$. Since their domains do not coincide, the functions f and g are different.

In logic, one considers special sets whose elements are called modalities or degrees of truth. In classical logic, such a set consists of two elements, *truth* and *falsity*. In the sequel, truth will be denoted by the symbol T and falsity by F.

Let A be any set. An n-ary function f defined on A with values in the set $\{\mathsf{T}, \mathsf{F}\}$ is said to be an *n-ary predicate on A*. The collection of sequences $(a_1, \ldots, a_n)$ in A^n for which

$$f(a_1, \ldots, a_n) = \mathsf{T}$$

is termed the (n-ary) relation in A corresponding to predicate f. Conversely, let $\alpha \subseteqq A^n$ be a given n-ary relation in A. Defining

$$f(a_1, \ldots, a_n) = \begin{cases} \mathsf{T} \text{ if } (a_1, \ldots, a_n) \in \alpha, \\ \mathsf{F} \text{ if } (a_1, \ldots, a_n) \notin \alpha, \end{cases}$$

we obtain an n-ary predicate corresponding to relation α. Thus, n-ary predicates and n-ary relations in an arbitrary set are in one-to-one correspondence.

The *characteristic function* of an n-ary relation α in set A is an n-ary function χ from A^n into N defined by

$$\chi(a_1, \ldots, a_n) = \begin{cases} 1, & \text{if } (a_1, \ldots, a_n) \in \alpha, \\ 0, & \text{if } (a_1, \ldots, a_n) \notin \alpha. \end{cases}$$

The *partial characteristic function* of an n-ary relation $\alpha \subseteq A^n$ is the partial n-ary function defined by

$$\chi(a_1, \ldots, a_n) = \begin{cases} 1, & \text{if } (a_1, \ldots, a_n) \in \alpha, \\ \text{indeterminate} & \text{if } (a_1, \ldots, a_n) \notin \alpha. \end{cases}$$

For example, if φ and ψ are respectively the characteristic function and partial characteristic function for the relation $<$ on N, then

$$\varphi(1, 3) = \psi(1, 3) = 1, \qquad \varphi(1, 1) = 0,$$

$$\psi(1, 1) = \text{indeterminate}.$$

The subsets of a set A are unary relations in A. Therefore the *characteristic function of a subset* is a function of a single variable equaling 1 at the points of the subset and 0 at points not in the subset. The partial characteristic function of a subset equals 1 at points of the subset and is indeterminate exterior to the subset.

In the sequel, we often shall not distinguish between a relation and the predicate P corresponding to it. Instead of $P(x_1, \ldots, x_m) = \mathsf{T}$, we shall write the predicate simply as $P(x_1, \ldots, x_m)$. In particular, we shall sometimes write the equality

$$P(x, y) = Q(x, y)$$

of predicates P and Q in the relation form

$$P(x, y) \Leftrightarrow Q(x, y).$$

By definition, each partial n-ary operation F on a set A corresponds to a subset of the cartesian power A^{n+1} comprising the sequences $(a_1, \ldots, a_n, a_{n+1})$ in A^{n+1} for which the value $F(a_1, \ldots, a_n)$ of F is defined and equal to a_{n+1}. This subset is sometimes called the *graph* of the partial operation F and one speaks for example, of the characteristic function of the graph of the partial operation instead of the characteristic function of the partial operation itself. We shall accept this point of view and let both expressions stand for the same thing.

Sometimes one has to consider 0-ary (nullary) operations and 0-ary predicates. A 0-ary operation on a set A is a fixed element of the set and a 0-ary predicate is truth or falsity.

The arity of an operation F will be denoted by $n(F)$ and that of a predicate P by $n(P)$.

2.2 Algebraic Systems. Let α and β be two ordinal numbers. A *type* τ *of order* (α, β) is a pair of mappings of the sets $W(\alpha)$ and $W(\beta)$ into $N = \{0, 1, 2, \ldots\}$: $W(\alpha) \to N$ and $W(\beta) \to N$. A type τ will be written as

$$\tau = \langle m_0, \ldots, m_\xi, \ldots; n_0, \ldots, n_\eta, \ldots \rangle$$

$$(\xi < \alpha,\ \eta < \beta).$$

Two types τ and τ' will be regarded *equal* if and only if each is of the same order (α, β), $m_\xi = m_\xi'$ for all $\xi < \alpha$ and $n_\eta = n_\eta'$ for all $\eta < \beta$. A type is said to be *finite* if the α and β determining its order (α, β) are both finite.

An *algebraic system* (or simply *system*) of type τ is a triple $\mathfrak{A} = \langle A, \Omega_F, \Omega_P \rangle$ consisting of a nonempty set A, a set of operations $\Omega_F = \{F_0, \ldots, F_\xi, \ldots\}$ on A for each $\xi < \alpha$ and a set of predicates $\Omega_P = \{P_0, \ldots, P_\eta, \ldots\}$ on A for each $\eta < \beta$ and such that the arities of the operations and predicates satisfy: 1) $n(F_\xi) = m_\xi$ for all $\xi < \alpha$ and 2) $n(P_\eta) = n_\eta$ for all $\eta < \beta$. Ω_F is called the *operator domain* and Ω_P the *predicate domain.*

The set A is the *carrier* or *underlying set* of system $\mathfrak{A}$ and its elements are the elements of the system. The *cardinality* or *order* of $\mathfrak{A}$ is taken to be $|A|$, the cardinality of A, and is also denoted by $|\mathfrak{A}|$.

In contrast to other operations and predicates that may be defined on set A, the operations $F_\xi (\xi < \alpha)$ and predicates $P_\eta (\eta < \beta)$ are said to be *basic* or *primitive.* The values of basic 0-ary operations in a system are said to be *basic* or *distinguished elements* of the system.

If we take the union of Ω_F and Ω_P for system $\mathfrak{A}$ and we let $\Omega = \Omega_F \cup \Omega_P$, we can write $\mathfrak{A}$ in the condensed form $A = \langle A, \Omega \rangle$.

Let $\mathfrak{A} = \langle A, \Omega \rangle$ and $\mathfrak{B} = \langle B, \Omega' \rangle$ be two given systems of the same type τ of order (α, β). Then the basic operations $F_\xi \in \Omega$ and $G_\xi \in \Omega'$ for each $\xi < \alpha$ and the basic predicates $P_\eta \in \Omega$ and $Q_\eta \in \Omega'$ for each $\eta < \beta$ are said to be *like.* Like operations and predicates in systems $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \ldots$ of the same type (*equitype* systems) will often be symbolized identically, say, $F_0, \ldots, F_\xi, \ldots, P_0, \ldots, P_\eta, \ldots (\xi < \alpha,\ \eta < \beta)$.

A system $\mathfrak{A} = \langle A, \Omega \rangle$ is *finite* if A is a finite set.

A system $\mathfrak{A}$ whose type is of finite order will be expressed as $\mathfrak{A} = \langle A; F_0, \ldots, F_{s-1}; P_0, \ldots, P_{t-1} \rangle$ or $\langle A; F_1, \ldots, F_s; P_1, \ldots, P_t \rangle$.

An algebraic system $\mathfrak{A} = \langle A, \Omega \rangle$ is called an *algebra* if $\Omega_P = \varnothing$ and a *model* (or *relational system*) if $\Omega_F = \varnothing$.

Examples of algebraic systems are

$$\mathfrak{A} = \langle Z; + \rangle,$$

$$\mathfrak{B} = \langle R; +, -, \times \rangle,$$

$$\mathfrak{C} = \langle Z; +; \leqq \rangle,$$

$$\mathfrak{D} = \langle Z; \leqq \rangle,$$

where Z is the set of integers, R is the set of rationals and $+$, $-$, and $\times$ are the ordinary operations of addition, subtraction and multiplication. According to the above, $\mathfrak{A}$ and $\mathfrak{B}$ are algebras of type $\langle 2 \rangle$ and $\langle 2, 2, 2 \rangle$, respectively, $\mathfrak{C}$ is an algebraic system of type $\langle 2; 2 \rangle$ and $\mathfrak{D}$ is a model of type $\langle 2 \rangle$. The algebraic system $\mathfrak{A} = \langle N; S, \times, 0, 1 \rangle$, where N is the set of all natural numbers and $S(x) = x + 1$, is an algebra of type $\langle 1, 2, 0, 0 \rangle$ with two distinguished elements 0 and 1.

Let us give a few further examples of algebras and models. Let U be the collection of circles and V the collection of convex closed polygons in the plane. The algebra and model,

$$\mathfrak{B} = \langle V; \cap \rangle, \qquad \mathfrak{U} = \langle U; \subseteq \rangle$$

have been encountered in the literature. The algebra

$$\left\langle W; +, -, \times, \frac{d}{dz} \right\rangle,$$

where W is the set of functions of a complex variable z analytic in a given domain, has also been considered by authors.

If the word "operation" is replaced in the definition of algebraic system by "partial operation", we obtain a *partial system.*

Each m-ary operation F defined on a set A is an $(m + 1)$-ary relation. Let P denote the corresponding $(m + 1)$-ary predicate for which

$$P(x_1, \ldots, x_m, y) \Leftrightarrow F(x_1, \ldots, x_m) = y \quad (x_1, \ldots, x_m, y \in A).$$

If the operations F_ξ in algebraic system $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_{\alpha+\eta}\} \rangle$ $(\xi < \alpha, \eta < \beta)$ are replaced by corresponding predicates P_ξ, we obtain a model

$$\mathfrak{A}^* = \langle A, \{P_\xi\} \rangle \qquad (\zeta < \alpha + \beta),$$

which is the model *representing* the algebraic system $\mathfrak{A}$. A model representing a system $\langle A, \Omega \rangle$ will be written in the abridged form $\mathfrak{A}^* = \langle A, \Omega_M \rangle$.

For example, in order to represent the algebra $\langle Z; + \rangle$ by a model, it suffices to introduce the predicate

$$S(x, y, z) \Leftrightarrow x + y = z,$$

representing the operation of addition. The model $\langle Z; S \rangle$ then represents the algebra $\langle Z; + \rangle$. In view of the simplicity of going from an algebra to its corresponding model, one often introduces no new notation and one speaks of the model $\langle Z; + \rangle$ meaning by this the model $\langle Z; S \rangle$. Nevertheless, it should be understood that certain notions to be defined later on have a different meaning for algebra $\langle Z; + \rangle$ than for model $\langle Z; S \rangle$.

By a mapping of an algebraic system $\mathfrak{A}$ into an algebraic system $\mathfrak{B}$, we shall mean a mapping of the carrier A of $\mathfrak{A}$ into the carrier B of $\mathfrak{B}$.

An *isomorphism* of an algebraic system $\mathfrak{A} = \langle A; F_0, \ldots, F_\xi, \ldots, P_0, \ldots, P_\eta, \ldots \rangle$ $(\xi < \alpha, \ \eta < \beta)$ of type τ into an algebraic system $\mathfrak{B} = \langle B; G_0, \ldots, G_\xi, \ldots, Q_0, \ldots, Q_\eta, \ldots \rangle$ of the same type is a one-to-one mapping φ of $\mathfrak{A}$ into $\mathfrak{B}$ that preserves basic operations and predicates, i.e., satisfying

$$F_\xi(x_1, \ldots, x_{m_\xi})\varphi = G_\xi(x_1\varphi, \ldots, x_{m_\xi}\varphi), \tag{1}$$

$$P_\eta(x_1, \ldots, x_{n_\eta}) \Leftrightarrow Q_\eta(x_1\varphi, \ldots, x_{n_\eta}\varphi) \tag{2}$$

for all $x_1, x_2, \ldots$ in A and for all $\xi < \alpha$ and $\eta < \beta$.

An isomorphism φ of a system $\mathfrak{A}$ is said to be *onto* an equitype system $\mathfrak{B}$ if the carrier A is mapped onto the carrier B. An isomorphism of a system $\mathfrak{A}$ onto itself is called an *automorphism.*

A *homomorphism* of a system $\mathfrak{A}$ into an equitype system $\mathfrak{B}$ is a mapping φ of $\mathfrak{A}$ into $\mathfrak{B}$ satisfying condition (1) and the condition

$$P_\eta(x_1, \ldots, x_{n_\eta}) \Rightarrow Q_\eta(x_1\varphi_1, \ldots, x_{n_\eta}\varphi) \ (x_1, x_2, \ldots \in A; \ \eta < \beta). \tag{3}$$

It is apparent from this that every isomorphism is a homomorphism. That every one-to-one homomorphism is an isomorphism is in general false. For example, let

$$\mathfrak{A} = \langle N; \sigma \rangle, \qquad \mathfrak{B} = \langle N; < \rangle,$$

σ being a binary relation on the nonnegative integers N which is false for any pair of such numbers. The models $\mathfrak{A}$ and $\mathfrak{B}$ are of the same type. Each mapping of $\mathfrak{A}$ into $\mathfrak{B}$ is a homomorphism. For, condition (3) is satified automatically here since the relation $x\sigma y$ is always false and hence the implication (3) is true. It is unnecessary to verify condition (1) because there are no operations in the systems. At the same time, there

are no isomorphisms of $\mathfrak{A}$ into $\mathfrak{B}$. Indeed, if φ were an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, the numbers 0φ and 1φ would be distinct. Hence either $0\varphi < 1$ or $1\varphi < 0$. Condition (2) would imply that either $0\sigma 1$ or $1\sigma 0$ is true contrary to the fact that $x\sigma y$ is false for any x and y in N.

Observe however that *every homomorphism φ of a finite system $\mathfrak{A} = \langle A, \Omega\rangle$ onto itself is an isomorphism.*

To see this, let $a_1, a_2, \ldots, a_n$ be elements of A for which $P(a_1\varphi, a_2\varphi, \ldots, a_n\varphi)$ $(P \in \Omega)$ is a true relation. By condition (3) in the definition of homomorphism, $P(a_1\varphi^k, a_2\varphi^k, \ldots, a_n\varphi^k)$ is also a true relation for each $k = 2, 3, \ldots$ But φ is a one-to-one mapping of a finite set A onto itself. Thus, for some k, φ^k is the identity mapping implying that $P(a_1, a_2, \ldots, a_n) = \mathsf{T}$.

Isomorphism and homomorphism are two of the basic concepts of the theory of algebraic systems. It is therefore worth noting the special case in which models $\mathfrak{A} = \langle A, \{P_\eta\}\rangle$ and $\mathfrak{B} = \langle A, \{Q_\eta\}\rangle$ $(\eta < \beta)$ have the same carrier A. By (3), the identity mapping of A onto A is then a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ if and only if the basic relations satisfy

$$P_\eta \subseteqq Q_\eta \qquad (\eta < \beta).$$

The identity mapping of A onto A is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ if and only if

$$P_\eta = Q_\eta \qquad (\eta < \beta).$$

A comparison of conditions (2) and (3) also shows that a one-to-one mapping φ of an algebraic system $\mathfrak{A}$ onto an equitype algebraic system $\mathfrak{B}$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ if and only if φ and φ^{-1} are homomorphisms of $\mathfrak{A}$ onto $\mathfrak{B}$ and $\mathfrak{B}$ onto $\mathfrak{A}$, respectively. In particular, if φ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then φ^{-1} is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$.

Let φ be a homomorphism of an algebraic system $\mathfrak{A}$ into an algebraic system $\mathfrak{B}$ and let ψ be a homomorphism of $\mathfrak{B}$ into a system $\mathfrak{C}$. It is easy to show that the mapping $\varphi\psi$ of $\mathfrak{A}$ into $\mathfrak{C}$ satisfies conditions (1) and (3) and so *a composition of homomorphisms is a homomorphism.*

By considering inverse mappings, we obtain the result that *a composition of isomorphisms of algebraic systems is an isomorphism.*

In contrast to models, *a one-to-one homomorphism of an algebra onto an algebra is an isomorphism.*

This is an immediate consequence of our definitions since isomorphisms and homomorphisms of algebras are characterized by the fulfillment of the same identities (1).

An algebraic system $\mathfrak{A}$ is said to be *isomorphic* to system $\mathfrak{B}$ if there exists an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

The above implies that *isomorphism between algebraic systems is a reflexive symmetric transitive relation.* Therefore all algebraic systems

of given type can be partitioned into classes of mutually isomorphic systems (*isoclasses*). The theory of algebraic systems is primarily the study of the properties of algebraic systems that are preserved under isomorphism and are thus identical for all isomorphic systems. Such properties are often said to be the *abstract* properties of systems. Here, the abstract properties of a system are the properties of its basic operations and predicates that do not depend on the nature of the elements comprising the system. Examples of the simplest abstract properties would be type and cardinality of a system since type and cardinality of isomorphic systems are necessarily identical.

At times one has to construct an algebraic system isomorphic to a given system having a different underlying set. We shall present the simplest procedure often used to solve this problem.

Let there be given an algebraic system $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ $(\xi < \alpha, \eta < \beta)$, a set B and a one-to-one onto-mapping $\varphi: A \to B$. It is required to construct an algebraic system $\mathfrak{B}$ with carrier B such that φ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

The solution of this problem is trivial. We define operations $G_\xi (\xi < \alpha)$ and predicates $Q_\eta (\eta < \beta)$ on B setting

$$G_\xi(y_1, \ldots, y_{m_\xi}) = F_\xi(y_1\varphi^{-1}, \ldots, y_{m_\xi}\varphi^{-1})\varphi,$$

$$Q_\eta(y_1, \ldots, y_{n_\eta}) = P_\eta(y_1\varphi^{-1}, \ldots, y_{n_\eta}\varphi^{-1}).$$

The mapping φ is clearly an isomorphism of $\mathfrak{A}$ onto the resultant algebraic system $\mathfrak{B} = \langle B, \{G_\xi\}, \{Q_\eta\}\rangle$.

At the outset of this subsection, each algebraic system $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ $(\xi < \alpha, \eta < \beta)$ was assigned a uniquely determined model $A^* = \langle A, P_\zeta\rangle$ $(\zeta < \alpha + \beta)$ representing system $\mathfrak{A}$. Let $\mathfrak{B} = \langle B, \{G_\xi\}, \{Q_\eta\}\rangle$ be some other system of the same type as $\mathfrak{A}$, and let $\mathfrak{B}^* = \langle B, Q_\zeta\rangle$ $(\zeta < \alpha + \beta)$ be the model representing $\mathfrak{B}$. Suppose that φ is a given homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Will the mapping φ be a homomorphism of the *model* $\mathfrak{A}^*$ into the model $\mathfrak{B}^*$? The obviously affirmative answer is contained in our next theorem.

Theorem 1. *A mapping φ of an algebraic system $\mathfrak{A}$ into an algebraic system $\mathfrak{B}$ is a homomorphism if and only if φ is a homomorphism of the model $\mathfrak{A}^*$ representing $\mathfrak{A}$ into the model $\mathfrak{B}^*$ representing $\mathfrak{B}$.*

To this end, let φ be a homomorphism of model $\mathfrak{A}^*$ into model $\mathfrak{B}^*$. Let F_ξ and G_ξ be like basic m-ary operations in $\mathfrak{A}$ and $\mathfrak{B}$ and let P_ξ and Q_ξ be the corresponding predicates in models $\mathfrak{A}^*$ and $\mathfrak{B}^*$. For any elements $a_1, \ldots, a_m$ in the carrier of $\mathfrak{A}$, consider $a = F_\xi(a_1, \ldots, a_m)$. We then have

$$P_\xi(a_1, \ldots, a_m, a) = \mathsf{T}.$$

Hence

$$Q_\xi(a_1\varphi, \ldots, a_m\varphi, a\varphi) = \mathsf{T}$$

or in other words,

$$G_\xi(a_1\varphi, \ldots, a_m\varphi) = a\varphi,$$

as required. The converse statement is proved in the same way.

In conclusion, we give a familiar example of an isomorphism and of a homomorphism in the arithmetic of real numbers. Let R be the set of all real numbers, R_+ the set of positive reals and R_0 the set of nonnegative reals. The formula

$$\log(a \cdot b) = \log a + \log b$$

shows that the mapping $a\varphi = \log a$ is an *isomorphism* of the algebra $\langle R_+; \cdot\rangle$ onto the algebra $\langle R; +\rangle$.

On the other hand, from the formula

$$|x \cdot y| = |x| \cdot |y|$$

it follows that the mapping $x \to |x|$ is a *homomorphism* of the algebra $\langle R, \cdot\rangle$ onto the algebra $\langle R_0; \cdot\rangle$. The mapping $x \to |x|$ is not one-to-one since each number in R_0 other than 0 has two pre-images in R.

2.3 Subsystems. Generating Sets. A nonempty subset A_1 of the carrier A of an algebraic system $\mathfrak{A} = \langle A, \Omega\rangle$ is said to be *closed in* $\mathfrak{A}$ if A_1 is closed under each basic operation F_ξ in the system, or in other words, if the application of each F_ξ to elements in A_1 yields elements in A_1. Let A_1 be closed in $\mathfrak{A}$ and let F_ξ^* and P_η^* be the respective restrictions to A_1 of operation F_ξ and predicate P_η in Ω. This results in an algebraic system $\mathfrak{A}_1 = \langle A_1, \Omega^*\rangle$ called a *subsystem* of A. If $\mathfrak{A}$ is an algebra or model, $\mathfrak{A}_1$ is a *subalgebra* or *submodel* of $\mathfrak{A}$.

A subsystem $\mathfrak{A}_1$ is uniquely determined by the subset A_1. Therefore instead of "subsystem $\langle A_1, \Omega^*\rangle$", one often writes "subsystem $\langle A_1, \Omega\rangle$" or simply "subsystem A_1".

If $\mathfrak{A}$ is a model, each nonempty subset A_1 of $\mathfrak{A}$ will be closed. Thus, *each nonempty subset of the carrier of a model is a submodel.*

Section 2.2 showed how to construct a model $\mathfrak{M} = \langle A, \Omega_M\rangle$ representing an algebraic system $\mathfrak{A} = \langle A, \Omega\rangle$ in predicate form. It is clear that each subsystem A_1 of $\mathfrak{A}$ is representable by submodel A_1 of $\mathfrak{M}$. But not every submodel A_1 of $\mathfrak{M}$ represents a subsystem of $\mathfrak{A}$. In fact, each nonempty subset A_1 of A is a submodel of $\mathfrak{M}$, whereas only the closed subsets in $\mathfrak{A}$ are subsystems. The submodels of $\mathfrak{M}$ are often said to be *submodels of* $\mathfrak{A}$. The *closed* submodels of A are then identified with the corresponding *subsystems* of $\mathfrak{A}$. The distinction between a submodel and a subalgebra is well clarified by the following example.

Let the carrier of algebra $\mathfrak{A}$ consist of the vertices a_1, a_2, a_3, a_4, a_5 and a_6 of a regular hexagon (Fig. 1). Consider the operation F defined by

$$F(a_i) = a_{i+1} \quad (i \neq 6), \quad F(a_6) = a_1.$$

The algebra $\langle A, F\rangle$ has no subalgebras other than itself since on applying F to a_i and then taking $F^2(a_i) = F(F(a_i)), \ldots, F^5(a_i)$, we obtain all

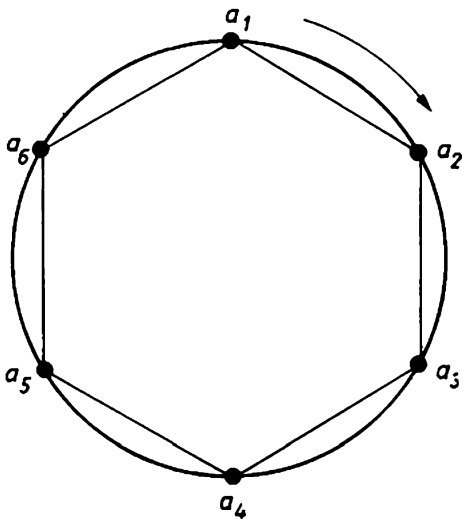

Fig. 1

six vertices. Yet the algebra $\mathfrak{A}$ has $2^6 - 1 = 63$ submodels. For example, $\mathfrak{A}$ contains the submodel

$$\langle\{a_1, a_2, a_5, a_6\};\ \{(a_1, a_2), (a_5, a_6), (a_6, a_1)\}\rangle.$$

Observe that if there are 0-ary operations among the basic operations, then each subsystem will contain all distinguished elements of the system. One of the most important properties of subsystems is contained in the following theorem.

Theorem 1. *The intersection of a family of subsystems of an algebraic system* $\mathfrak{A} = \langle A, \Omega\rangle$ *is either empty or is a subsystem.*

It is necessary to show that the intersection D of a family $\{A_\alpha \mid \alpha \in T\}$ of closed subsets of A is either empty or is closed. Suppose $D \neq \varnothing$. If we perform a basic operation F_ξ on elements $a_1, \ldots, a_{m_\xi}$ in D, we obtain some element $a \in A$. Since each set A_α is closed under F_ξ and contains $a_1, \ldots, a_{m_\xi}$, it contains a and so $a \in D$, as required.

Suppose there are 0-ary operations among the basic operations. Then the distinguished elements of the system, being in each subsystem, are contained also in the intersection of every family of subsystems. Thus, *for systems with distinguished elements, the intersection of a family of subsystems is again a subsystem.* In particular, the intersection of all subsystems of the system is a subsystem. It is the *minimal* or *principal* subsystem of the system.

However a system having no 0-ary basic operations may also have a minimal subsystem. Consider for example the algebra

$$\mathfrak{Z} = \langle \{0, \pm 1, \pm 2, \ldots\}; +, -, \times \rangle,$$

the ring of whole rational numbers. Each subalgebra is closed under subtraction. Therefore along with a number a it contains the number $a - a = 0$ also. On the other hand, the set $\{0\}$ is itself a subalgebra of $\mathfrak{Z}$. Hence it follows that the subalgebra $\{0\}$ is minimal in $\mathfrak{Z}$.

By contrast, the algebra

$$\langle \{1, 2, 3, \ldots,\}; + \rangle,$$

is easily seen to have no minimal subalgebra.

Consider again an arbitrary algebraic system $\mathfrak{A} = \langle A, \Omega \rangle$. Let B be a nonempty subset of A. Among the subsystems of $\mathfrak{A}$ are necessarily ones that contain B. For instance, B is contained in the subsystem comprising the entire system $\mathfrak{A}$. Let $\mathfrak{B}$ denote the intersection of all subsystems containing B. By Theorem 1, $\mathfrak{B}$ is a subsystem. It contains B and is contained in each subsystem containing B. In other words, $\mathfrak{B}$ is the minimal subsystem containing B. $\mathfrak{B}$ is said to be the subsystem of $\mathfrak{A}$ *generated* by B and the elements of B are sometimes termed the *generating elements* of $\mathfrak{B}$. If the subsystem $\mathfrak{B}$ coincides with $\mathfrak{A}$, then B is said to be a *generating set* for $\mathfrak{A}$.

We point out right away that different sets may generate the same subsystem. For example, in the algebra depicted in Fig. 1, each element generates the whole algebra. By contrast, no element of the algebra $\langle N, \times \rangle$ $(N = 1, 2, 3, \ldots)$ individually generates the whole algebra $\langle N, \times \rangle$. In addition, it is easy to show that no finite set of elements of this algebra can generate the whole algebra.

We mention the following further properties of generating sets which are immediate consequences of our definitions. For brevity, let $[C]_{\mathfrak{A}}$ denote the subsystem generated in $\mathfrak{A}$ by set C. We have

1. $C \subseteq [C]_{\mathfrak{A}}$.
2. $C \subseteq D \Rightarrow [C]_{\mathfrak{A}} \subseteq [D]_{\mathfrak{A}}$.
3. $[C]_{\mathfrak{A}} \subseteq \left[[C]_{\mathfrak{A}}\right]_{\mathfrak{A}}$.
4. $[C]_{[C]_{\mathfrak{A}}} = [C]_{\mathfrak{A}}$.

It is easy to see that the union of two subsystems need not be a subsystem. Nevertheless, we can supplement Theorem 1 by a proposition formulated as Theorem 2 below.

Let A be an arbitrary set and let B be a subset of A. A family $\mathfrak{S} = \{A_\alpha \mid \alpha \in T\}$ of subsets A_α of A is said to be *local* **in** B if each finite subset of B is a subset of some A_α in S. A family S local in its own union, i.e., local in the set $B = \bigcup_{\alpha \in T} A_\alpha$, is simply said to be local.

Theorem 2. *The union of a nonempty local family* $\mathfrak{S}$ *of subsystems of an algebraic system* $\mathfrak{A}$ *is a subsystem of* $\mathfrak{A}$.

To show this, let F_ξ be any basic operation of $\mathfrak{A}$ and let $a_1, \ldots, a_{m_\xi}$ be elements in the union. By assumption, the family $\mathfrak{S}$ has a subsystem A_α containing $a_1, \ldots, a_{m_\xi}$ and therefore also containing the element $F_\xi(a_1, \ldots, a_{m_\xi})$. Both the element $F_\xi(a_1, \ldots, a_{m_\xi})$ and the system A_α are contained in the union and so the union is closed under each operation.

Theorem 2 implies, in particular, that *the union of an ascending chain*

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \ldots \subset \mathfrak{A}_n \subset \ldots$$

of subsystems of an algebraic system $\mathfrak{A}$ *is a subsystem of* $\mathfrak{A}$.

We now consider the behavior of subsystems under homomorphic mappings.

Theorem 3. *Under homomorphism of one algebraic system into another, the image of a subsystem and any nonempty complete pre-image of a subsystem are subsystems.*

To prove this, let α be a homomorphism of $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ into $\mathfrak{B} = \langle B, \{G_\xi\}, \{Q_\eta\}\rangle$ and let C be a subsystem of $\mathfrak{A}$, $C \subseteq A$.

Take an arbitrary basic operation G_ξ in $\mathfrak{B}$ and arbitrary elements $b_1, \ldots, b_{m_\xi}$ in the image C^α of subsystem C. Let $a_1, \ldots, a_{m_\xi}$ be pre-images of $b_1, \ldots, b_{m_\xi}$ in C and let $F_\xi(a_1, \ldots, a_{m_\xi}) = a$. From (1) of Section 2.2 we obtain

$$G_\xi(b_1, \ldots, b_{m_\xi}) = F_\xi(a_1, \ldots, a_{m_\xi})\alpha = a\alpha.$$

Since $a \in C$, $a\alpha \in C^\alpha$, or in other words, the set C^α is closed under basic operations of system $\mathfrak{B}$. And so the first statement of Theorem 3 is proved.

We proceed now to prove the second statement and we let $\mathfrak{D}$ be a subsystem of $\mathfrak{B}$. Suppose the set C of elements in $\mathfrak{A}$ which map into elements of $\mathfrak{D}$ under the homomorphism α is nonempty. We have to

show that C is closed. Choose a basic operation F_ξ and elements $a_1, \ldots, a_{m_\xi}$ in C. By definition of homomorphism,

$$F_\xi(a_1, \ldots, a_{m_\xi})\alpha = G_\xi(a_1\alpha, \ldots, a_{m_\xi}\alpha) \in \mathfrak{D},$$

and so $F_\xi(a_1, \ldots, a_{m_\xi}) \in C$, as required.

The following special case of Theorem 3 is worth noting: *Under a homomorphism α of system $\mathfrak{A}$ into system $\mathfrak{B}$, the image of $\mathfrak{A}$ is a subsystem of $\mathfrak{B}$ and α is a homomorphism of $\mathfrak{A}$ onto the system $\mathfrak{A}^\alpha$.*

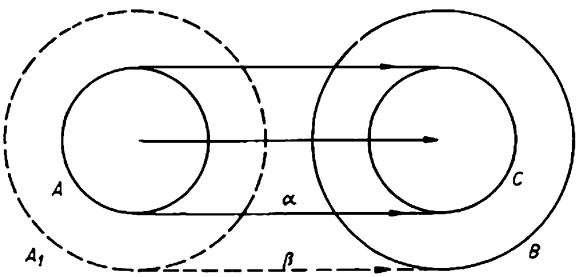

Fig. 2

Isomorphisms are special homomorphisms and so the assertions of Theorem 3 are also valid for isomorphisms.

To form a system satisfying certain conditions and containing a given system, one often makes use of the following very simple construction.

Let α be an isomorphism of an algebraic system $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ onto a subsystem $\mathfrak{C} = \langle C, \{G_\xi\}, \{Q_\eta\}\rangle$ of an algebraic system $\mathfrak{B} = \langle B, \{G_\xi\}, \{Q_\eta\}\rangle$ $(C \subseteq B)$. Does there exist an algebraic system $\mathfrak{A}_1$ of which $\mathfrak{A}$ is a subsystem and which is isomorphic to $\mathfrak{B}$? To solve this problem, we choose the set A_1 to be $(B - C) \cup A$ (Fig. 2) and we let β be the mapping of A_1 onto B coinciding with α on the subset A of A_1 and with the identity mapping on its subset $B - C$. β is clearly a one-to-one mapping of A_1 onto B.

Using β, we can transfer the structure of system $\mathfrak{B}$ to A_1 (see Section 2.2), i.e., we define predicates P_η and operations F_ξ on A_1 by means of the formulas

$$F_\xi(x_1, \ldots, x_{m_\xi}) = G_\xi(x_1\beta, \ldots, x_{m_\xi}\beta)\beta^{-1},$$

$$P_\eta(x_1, \ldots, x_{n_\eta}) = Q_\eta(x_1\beta, \ldots, x_{n_\eta}\beta).$$

As a result, we obtain an algebraic system $\mathfrak{A}_1 = \langle A_1, \{F_\xi\}, \{P_\eta\}\rangle$. $\mathfrak{A}$ is clearly a subsystem of $\mathfrak{A}_1$ and β is an isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{B}$ coinciding with the preassigned isomorphism α on $\mathfrak{A}$.

One says briefly that $\mathfrak{A}_1$ results from $\mathfrak{B}$ by way of identification of the elements of $\mathfrak{A}$ with their corresponding elements in $\mathfrak{B}$.

The above procedure for embedding one system in another leads to the following important construction. Let $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ be a given infinite sequence of equitype algebraic systems and for each system $\mathfrak{A}_i$, let there be given an isomorphism φ_i of $\mathfrak{A}_i$ into $\mathfrak{A}_{i+1}$. The systems $\mathfrak{A}_i$ are depicted in Fig. 3 as solid lines and the isomorphisms φ_i as dashed lines.

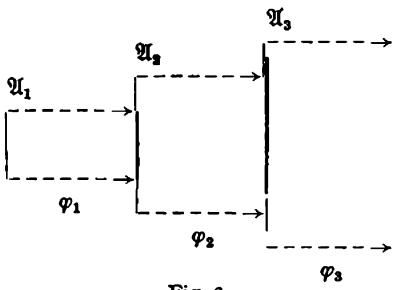

Fig. 3

On identifying $\mathfrak{A}_1$ with $\mathfrak{A}_1\varphi_1$, $\mathfrak{A}_2$ with $\mathfrak{A}_2\varphi_2$, etc., we obtain a sequence of embedded algebraic systems,

$$\mathfrak{A}_1 \subseteq \mathfrak{A}_2^* \subseteq \mathfrak{A}_3^* \subseteq \ldots \tag{1}$$

This means that if $\mathfrak{A}_k^* = \langle A_k, \{F_{k\xi}\}, \{P_{k\eta}\}\rangle$, $(\xi < \alpha,\ \eta < \beta)$, then

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots,$$

$$F_{1\xi} \subseteq F_{2\xi} \subseteq F_{3\xi} \subseteq \ldots \quad (\xi < \alpha),$$

$$P_{1\eta} \subseteq P_{2\eta} \subseteq P_{3\eta} \subseteq \ldots \quad (\eta < \beta).$$

Let A, F_ξ and P_η be the respective unions of sets A_k, $F_{k\xi}$ and $P_{k\eta}$. The relations F_ξ are clearly operations on A. Therefore the triple $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ $(\xi < \alpha,\ \eta < \beta)$ is an algebraic system containing the systems $\mathfrak{A}_1, \mathfrak{A}_2^*, \mathfrak{A}_3^*, \ldots$ as subsystems. The system $\mathfrak{A}$ is termed the *union* of the ascending chain of systems (1). The original sequence of isomorphisms

$$\mathfrak{A}_1 \xrightarrow{\varphi_1} \mathfrak{A}_2 \xrightarrow{\varphi_2} \mathfrak{A}_3 \xrightarrow{\varphi_3} \ldots \tag{2}$$

is called the *direct spectrum* and the system $\mathfrak{A}$ is said to be its *limit*.*)

The construction of the limit of direct spectrum (2) nowhere assumed that $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ are distinct systems. Of interest sometimes is the case

*) A spectrum of the form (2) is actually the simplest case of a direct spectrum [*Ed.*].

where not only all systems $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$ coincide but also all isomorphisms φ_i. Consider for example the algebra

$$\mathfrak{A}_1 = \langle \{1, 2, 3, \ldots\}; + \rangle.$$

The mapping φ: $a \to 2a$ is an isomorphic mapping of $\mathfrak{A}_1$ onto its subalgebra $\{2, 4, 6, \ldots\}$ and so the chain of isomorphisms

$$\mathfrak{A}_1 \xrightarrow{\varphi} \mathfrak{A}_1 \xrightarrow{\varphi} \mathfrak{A}_1 \xrightarrow{\varphi} \ldots \tag{3}$$

forms a direct spectrum. The question is: What is the limit algebra of the spectrum? In order to distinguish the elements of the $\mathfrak{A}_1$ located in the n-th position of the chain (3) from those of the algebra located in other positions, we shall denote the elements of the n-th algebra $\mathfrak{A}_1$ by the pairs $(n, 1), (n, 2), \ldots$ In accordance with the isomorphism φ, the pairs (n, i) and $(n + 1, 2i)$ must denote the same element in the limit algebra. This gives us the key to constructing the limit algebra $\mathfrak{A}$. Consider the set A of pairs (i, j) $(i, j = 1, 2, \ldots)$. The pairs (i, j) and $(i + k, 2^k j)$ will be called equivalent. The sum of (a, b) and (c, d) is defined to be the pair $(a + c, 2^c b + 2^a d)$ and any pair equivalent to it. The resultant algebra with the set $A/\sim$ as carrier, wherein $\sim$ stands for the specified equivalence, is then our required limit algebra. The properties of the limit algebra differ sharply from those of the original algebra. For instance, given the element $u = (a, b)$, the equation $x + x = u$ is solvable in the limit algebra. The solution is $x = (a + 1, b)$ since

$$(a + 1, b) + (a + 1, b) \sim (a + 1, 2b) \sim (a, b).$$

Hence it follows in particular that the limit algebra $\mathfrak{A}$ is not isomorphic to the original algebra $\mathfrak{A}_1$. For, if τ were an isomorphism of algebra $\mathfrak{A}$ onto algebra $\mathfrak{A}_1$, then from $x + x = u$ in $\mathfrak{A}$ it would follow that $x\tau + x\tau = u\tau$ in $\mathfrak{A}_1$. Therefore given any $a \in A_1$, we could find a solution x of $x + x = a\tau^{-1}$ in $\mathfrak{A}$ and then a solution $y = x\tau$ of $y + y = a$ in $\mathfrak{A}_1$. Since the equation $y + y = 1$ clearly has no solution in $\mathfrak{A}_1$, the algebras $\mathfrak{A}$ and $\mathfrak{A}_1$ are not isomorphic.

The mapping φ above was taken to be φ: $a \to 2a$. If instead one takes φ: $a \to 3a$, then the equation $x + x + x = u$ will be solvable for each u in the limit algebra, etc.

2.4 Congruences. Associated with each mapping φ of a set A onto a set B is the kernel equivalence σ_φ on A defined by (see Section 1.4)

$$x \sigma_\varphi y \Leftrightarrow x\varphi = y\varphi \qquad (x, y \in A).$$

If we set up a correspondence between each element $z \in B$ and its complete pre-image $z\varphi^{-1}$ in A, we obtain the one-to-one natural mappings

$$B \to A/\sigma_\varphi, \qquad A/\sigma_\varphi \to B.$$

Suppose now that A and B are the carriers of equitype algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ and that φ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. What additional properties does the kernel equivalence σ now possess? In order to answer this question, we shall have to introduce a few new concepts. These play an important part in many areas of algebra.

A relation $P(x_1, \ldots, x_n)$ on a set A is said to be *stable under an m-ary operation* F defined on A if for any elements $a_{i1}, a_{i2}, \ldots, a_{in}$ in A $(i = 1, 2, \ldots, m)$ the truth of the relations

$$P(a_{i1}, a_{i2}, \ldots, a_{in}) \ (i = 1, 2, \ldots, m)$$

implies the truth of the relation

$$P\big(F(a_{11}, \ldots, a_{m1}), \ldots, F(a_{1n}, \ldots, a_{mn})\big).$$

A relation P is *stable in an algebraic system* $\mathfrak{A}$ if it is stable under each basic operation in $\mathfrak{A}$.

In particular, a binary relation σ on A is stable under an operation F defined on A if any elements $a_1, a_1', \ldots, a_m\, a_m'$, in A connected by

$$\left.\begin{aligned} a_1 &\equiv a_1'(\sigma), \\ a_2 &\equiv a_2'(\sigma), \\ &\cdots\cdots \\ a_m &\equiv a_m'(\sigma), \end{aligned}\right\} \tag{1}$$

satisfy

$$F(a_1, \ldots, a_m) \equiv F(a_1', \ldots, a_m')\ (\sigma). \tag{2}$$

These definitions imply that an identically false relation on A or an identically true relation on A is stable under any operation defined on A.

An equivalence relation σ on an algebraic system $\mathfrak{A}$ is called a *congruence* on $\mathfrak{A}$ if σ is stable under each basic operation in $\mathfrak{A}$.

Let σ be a given congruence on $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ $(\xi < \alpha, \eta < \beta)$. We would like to convert the set A/σ of all σ-equivalence classes of A (see Section 1.4) into an algebraic system of same type as $\mathfrak{A}$. To this end, we need to define on A/σ operations $F_\xi{}^*$ and predicates $P_\eta{}^*$. Corresponding to each $a \in A$, let $[a]_\sigma$ denote the equivalence class containing a,

or in other words, the set of all elements in A comparable with a under σ. Given any $a_1, \ldots, a_{m_\xi}$ in A, define

$$F_\xi^*([a_1]_\sigma, \ldots, [a_{m_\xi}]_\sigma) = [F_\xi(a_1, \ldots, a_{m_\xi})]_\sigma. \tag{3}$$

If $[a_k]_\sigma = [a_k']_\sigma$ $(k = 1, \ldots, m_\xi)$, then the elements a_k and a_k' will satisfy the congruences (1) and so in consequence of (2) we have

$$F_\xi^*([a_1']_\sigma, \ldots, [a_{m_\xi}']_\sigma) = F_\xi^*([a_1]_\sigma, \ldots, [a_{m_\xi}]_\sigma).$$

This means that the operation F_ξ defined on A/σ through relation (3) is singlevalued.

Further, let $[a_1]_\sigma, \ldots, [a_{n_\eta}]_\sigma$ be arbitrary classes in A/σ. The relation $P_\eta^*([a_1]_\sigma, \ldots, [a_{n_\eta}]_\sigma]$ is defined to be true if *one can find* elements $a_k' \equiv a_k(\sigma)$ in the classes $[a_1]_\sigma, \ldots, [a_{n_\eta}]_\sigma$ for which $P_\eta(a_1', \ldots, a_{n_\eta}')$ is true in system $\mathfrak{A}$.

The triple $A/\sigma = \langle A/\sigma, \{F_\xi\}, \{P_\eta\}\rangle$ $(\xi < \alpha, \ \eta < \beta)$ is called the *quotient (algebraic) system of* $\mathfrak{A}$ *modulo congruence* σ.

The natural mapping $\varphi\colon a \to [a]_\sigma$ *of system* $\mathfrak{A}$ *onto the quotient system* A/σ *is a homomorphism for which* σ *is the kernel equivalence.*

Indeed, condition (3) implies

$$F_\xi(a_1, \ldots, a_{m_\xi})\varphi = [F_\xi(a_1, \ldots, a_{m_\xi})]_\sigma = F_\xi^*(a_1\varphi, \ldots, a_{m_\xi}\varphi),$$

and $P_\eta(a_1, \ldots, a_{n_\eta}) = \mathsf{T}$ implies $P_\eta^*(a_1\varphi, \ldots, a_{n_\eta}\varphi) = \mathsf{T}$, as required.

Let φ be a homomorphism of $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ onto $\mathfrak{B} = \langle B, \{G_\xi\}, \{Q_\eta\}\rangle$. If for any elements $b_1, \ldots, b_{n_\eta} \in B$ and for each basic predicate Q_η it follows from $Q_\eta(b_1, \ldots, b_{n_\eta}) = \mathsf{T}$ that there exist preimages $a_1, \ldots, a_{n_\eta}$ of $b_1, \ldots, b_{n_\eta}$ in A such that $P_\eta(a_1, \ldots, a_{n_\eta}) = \mathsf{T}$, then φ is called *strong*.

Since an algebra has no basic predicates, the concepts of homomorphism and strong homomorphism coincide in an algebra. In the case of models, there can be homomorphisms that are not strong.*)

An immediate consequence of the way a basic predicate is defined in a quotient system A/σ is that the natural mapping of A onto $\mathfrak{A}/\sigma$ is a strong homomorphism.

Theorem 1 (Homomorphism Theorem). *The kernel equivalence* σ *for each homomorphism* φ *of an algebraic system* $\mathfrak{A} = \langle A, \{F_\xi\}, \{P_\eta\}\rangle$ *onto*

*) For example, let $\mathfrak{A} = \langle\{a\}, P\rangle$ be a system consisting of a one-element carrier and a unary predicate P such that $P(a) = \mathsf{F}$. Let $\mathfrak{B} = \langle\{b\}, Q\rangle$ be another one-element system with unary predicate Q such that $Q(b) = \mathsf{T}$. The mapping $\varphi: a \to b$ is a homomorphism (and even one-to-one) but it is not strong since φ ist not an isomorphism between the models [*Ed.*].

an equitype system $\mathfrak{B} = \langle B, \{G_\xi\}, \{Q_\eta\}\rangle$ $(\xi < \alpha,\ \eta < \beta)$ *is a congruence on* $\mathfrak{A}$ *and the natural mapping* $\tau: \mathfrak{A}/\sigma \to \mathfrak{B}$ *is a homomorphism. If* φ *is a strong homomorphism, then the natural mapping of* $\mathfrak{A}/\sigma$ *onto* $\mathfrak{B}$ *is an isomorphism.*

Indeed, from (1) follows

$$a_k\varphi = a_k'\varphi \qquad (k = 1, \ldots, m_\xi).$$

Hence

$$G_\xi(a_1\varphi, \ldots, a_{m_\xi}\varphi) = G_\xi(a_1'\varphi, \ldots, a'_{m_\xi}\varphi),$$

i.e.,

$$F_\xi(a_1, \ldots, a_{m_\xi})\varphi = F_\xi(a_1', \ldots, a'_{m_\xi})\varphi,$$

and so (2) holds.

Now let $P_\eta^*([a_1], \ldots, [a_{n_\eta}])$ be a true relation for $a_1, \ldots, a_{n_\eta} \in A$. This means that for suitable $a_1', \ldots, a'_{n_\eta}$ in A we have

$$a_k\varphi = a_k'\varphi \qquad (k = 1, \ldots, n_\eta)$$

and $P_\eta(a_1', \ldots, a'_{n_\eta}) = \mathsf{T}$. The last relation implies that $Q_\eta(a_1'\varphi, \ldots, a'_{n_\eta}\varphi) = \mathsf{T}$. Since

$$[a]\tau = a\varphi \qquad (a \in A),$$

τ is a homomorphism.

Finally, let φ be a strong homomorphism and let $Q_\eta(b_1, \ldots, b_{n_\eta})$ be a true relation for some $b_1, \ldots, b_{n_\eta}$. Then the elements $b_1, \ldots, b_{n_\eta}$ have pre-images $a_1, \ldots, a_{n_\eta}$ in A such that $P_\eta(a_1, \ldots, a_{n_\eta}) = \mathsf{T}$. But in that event,

$$P_\eta^*([a_1], \ldots, [a_{n_\eta}]) = \mathsf{T}.$$

The kernel equivalence σ induced by the homomorphism φ of A will henceforth be referred to as the *kernel congruence.*

Theorem 1 shows that up to isomorphism the quotient systems of an algebraic system $\mathfrak{A}$ by its various congruences completely exhaust the set of all strongly homomorphic images of $\mathfrak{A}$. Therefore: a) *Finding (up to isomorphism) all strongly homomorphic images of a given algebraic system* $\mathfrak{A}$ and b) *Finding all congruences on* $\mathfrak{A}$ are equivalent problems.

Two congruences on every algebraic system $\mathfrak{A}$ are the trivial congruence ι coinciding with the equality relation in $\mathfrak{A}$ and the improper congruence $\bar{\omega}$ under which every two elements in $\mathfrak{A}$ are congruent to each other.*) Each ι-equivalence class consists of a single element and so the natural mapping of $\mathfrak{A}$ onto $\mathfrak{A}/\iota$ is an isomorphism. By contrast, the improper congruence combines all elements of $\mathfrak{A}$ into single class and $\mathfrak{A}/\bar{\omega}$ is therefore a system comprising a single element.

*) The congruence $\bar{\omega}$ is sometimes called *universal.* Any congruence on $\mathfrak{A}$ other than ι is said to be *nontrivial* and any other than $\bar{\omega}$, *proper* [*Trans.*].

Let us discuss a few examples. Consider the algebras

$$\mathfrak{A}_1 = \langle \{1, 2, 3, \ldots\}, + \rangle, \qquad \mathfrak{A}_2 = \langle \{-1, 1\}, \cdot \rangle$$

and let φ be a mapping of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ defined by $\varphi: a \to (-1)^a$. Since

$$(a + b)\varphi = (-1)^{a+b} = (-1)^a \cdot (-1)^b = a\varphi \cdot b\varphi,$$

φ is a homomorphism. The quotient algebra $\mathfrak{A}_1/\varphi$ consists of two classes. One is the class [1] of all positive odd integers and the other [2], the class of all even positive integers. It is easy to show that apart from the kernel congruence corresponding to the homomorphism φ, there are infinitely many other congruences on $\mathfrak{A}_1$.

Consider now a model $\langle A, P \rangle$ with a unary predicate P. We shall identify P with the set of elements on which it is true. The identity mapping of model $\langle A, P \rangle$ onto equitype model $\langle A, Q \rangle$ is a homomorphism if and only if $P \subseteq Q$. In particular, if $P = \varnothing$, the identity mapping of $\langle A, P \rangle$ onto any model $\langle A, Q \rangle$ is a homomorphism. Corresponding to all of these homomorphisms is the same trivial congruence "$=$" on A.

Before formulating our next theorem, we recall that if σ is a binary relation in A and $A_1 \subseteq A$, then σA_1 is the set of $x \in A$ for which A_1 has an element a_1 such that $x \sigma a_1 = \mathsf{T}$. The congruence $\bar{\omega}$ on A_1 defined above is the same as the cartesian square $A_1 \times A_1$ and so it is often written as $A_1{}^2$.

Theorem 2 (First Isomorphism Theorem). Let $\mathfrak{A}_1$ *be a subsystem of an algebraic system* $\mathfrak{A}$ *and let* σ *be a congruence on* $\mathfrak{A}$. *Then* $A_1{}^2 \cap \sigma$ *is a congruence on* A_1, σA_1 *is a subsystem of* $\mathfrak{A}$ *and the following one-to-one homomorphism holds*:

$$\mathfrak{A}_1/(\sigma \cap A_1{}^2) \to \sigma\mathfrak{A}_1/\sigma. \tag{4}$$

The arrow here designates a natural mapping under which each equivalence class of the quotient system on the left is carried into the equivalence class of the quotient system on the right that contains it.

The first statement of the theorem is trivial. The fact that the set σA_1 is closed under basic operations is also easy to establish. For, suppose F is a basic m-ary operation and $a_1, \ldots, a_m$ are arbitrary elements of σA_1. This means that A_1 contains elements $a_1', \ldots, a_m'$ related to $a_1, \ldots, a_m$ by the congruences (1). Since by hypothesis, $F(a_1', \ldots, a_m') \in A_1$, we conclude from (2) that $F(a_1, \ldots, a_m) \in \sigma A_1$.

Each $(\sigma \cap A_1{}^2)$-equivalence class of A_1 lies in some σ-equivalence class of σA_1 and each σ-equivalence class of σA_1 contains just one

$(\sigma \cap A_1^2)$-equivalence class of A_1. This fact is depicted schematically in Fig. 1; the sectors of the inner circle represent the $(\sigma \cap A_1^2)$-classes of A_1 while the sectors of the larger circle that contain them represent the corresponding σ-equivalence classes of A_1. Suppose for simplicity that system $\mathfrak{A}$ has a binary predicate P and that for u and v in $A_1/(\sigma \cap A_1^2)$, we have $P(u, v) = \mathsf{T}$. Then the classes u and v contain elements a and b such that $P(a, b) = \mathsf{T}$. But in that case, $P(a\sigma, b\sigma) = \mathsf{T}$ by the definition of quotient system. Thus the mapping (4) is a homomorphism with respect to all basic predicates. In a similar way, it can be

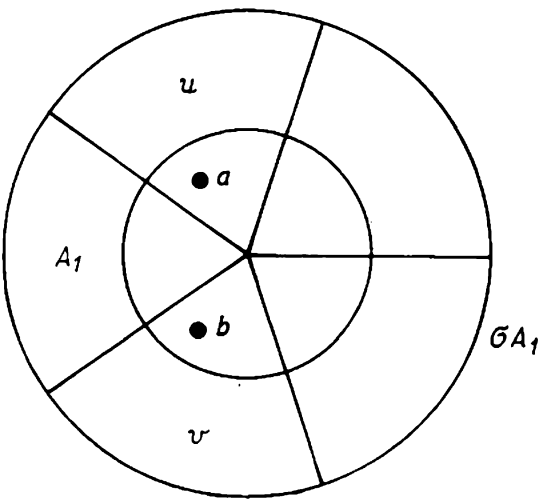

Fig. 1

shown that (4) is also a homomorphism with respect to all basic operations.

A one-to-one homomorphism of an algebra onto an algebra is an isomorphism. Therefore if $\mathfrak{A}$ is an algebra, the mapping (4) is an isomorphism.

Consider now the case where two congruences σ and η are given on the same algebraic system $\mathfrak{A}$ satisfying the condition $\sigma \subset \eta$. In Fig. 2, the elements of $\mathfrak{A}$ are represented by the points in the large square, the η-equivalence classes of $\mathfrak{A}$ by the four middle-sized squares and the σ-equivalence classes of $\mathfrak{A}$ by the sixteen little squares. The general situation is clearly similar. Namely, each η-equivalence class of $\mathfrak{A}$ splits up into several σ-equivalence classes and we can therefore assert that the congruence η induces an equivalence relation on the quotient

system $\mathfrak{A}/\sigma$, which we agree to denote by η/σ. The little squares are η/σ-equivalent if they lie in the same middle-sized square.*)

Theorem 3 (Second Isomorphism Theorem). *Let σ and η be congruences on algebraic system $\mathfrak{A}$ such that $\sigma \subset \eta$. Then the equivalence η/σ is a congruence on the quotient system $\mathfrak{A}/\sigma$ and the natural mapping*

$$\varphi\colon (\mathfrak{A}/\sigma)/(\eta/\sigma) \to \mathfrak{A}/\eta \tag{5}$$

is an isomorphism.

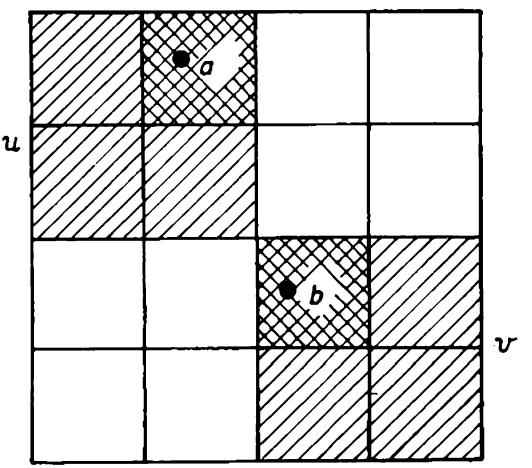

Fig. 2

The first statement is obvious and so we shall only prove the second one. Suppose for simplicity that the system $\mathfrak{A}$ has a binary predicate P and that for u and v in $\mathfrak{A}/\eta$ the predicate is true. This means that there exist $a \in u$ and $b \subset v$ for which $P(a, b) = \mathsf{T}$. This implies that $P([a]_\sigma, [b]_\sigma) = \mathsf{T}$ and so $P\big([[a]_\sigma]_{\eta/\sigma}, [[b]_\sigma]_{\eta/\sigma}\big) = \mathsf{T}$ in the quotient system $(\mathfrak{A}/\sigma)/(\eta/\sigma)$. In a similar way, one can show that from $P(x, y) = \mathsf{T}$ in the quotient system $(\mathfrak{A}/\sigma)/(\eta/\sigma)$ it follows that $P(x\varphi, y\varphi) = \mathsf{T}$ in the quotient system $\mathfrak{A}/\eta$.

We have shown that the mapping (5) is an isomorphism with respect to all basic predicates of $\mathfrak{A}$. The fact that it is also an isomorphism with respect to all basic operations in $\mathfrak{A}$ can be established in the same way.

---

*) η/σ, usually called a *fractional* equivalence, can be defined by the formula

$$[a]_\sigma \eta/\sigma [b]_\sigma \Leftrightarrow a \eta b,$$

where a and b are arbitrary elements of the given system [*Ed.*].

Section 1.4 considered various operations on equivalence relations. Let us see whether these operations result in congruences if the original equivalence relations are congruences.

Theorem 4. *The intersection σ of a family of congruences $\{\theta_\alpha\}$ on an algebraic system $\mathfrak{A}$ is a congruence on $\mathfrak{A}$.*

By Section 1.4, we know that σ is an equivalence on $\mathfrak{A}$. Consider some basic operation F on $\mathfrak{A}$ and let $a_1, a_1', \ldots, a_m, a_m'$ be elements of $\mathfrak{A}$ related by (1). Since σ is contained in each congruence θ_α, relation (2) holds true for θ_α. And since relation (2) is true for each congruence θ_α, it is true for their intersection σ.

Theorem 5. *The composition $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ of a finite chain of stable binary relations $\sigma_1, \sigma_2, \ldots, \sigma_k$ in an algebraic system $\mathfrak{A}$ is stable in $\mathfrak{A}$.*

It obviously suffices to prove the theorem for two factors. Let F be a basic m-ary operation on $\mathfrak{A}$ and let $a_1, a_1', \ldots, a_m, a_m'$ be elements of $\mathfrak{A}$ related by (1). By definition of composition of relations, one can find elements $b_1, \ldots, b_m$ in $\mathfrak{A}$ related by the congruences

$$\begin{aligned} a_1 &\equiv b_1(\sigma_1), \quad b_1 \equiv a_1'(\sigma_2), \\ a_2 &\equiv b_2(\sigma_1), \quad b_2 \equiv a_2'(\sigma_2), \\ &\ldots\ldots\ldots\ldots\ldots \\ a_m &\equiv b_m(\sigma_1), \quad b_m \equiv a_m'(\sigma_2). \end{aligned}$$

From the relations in the first column, we obtain

$$F(a_1, \ldots, a_m) \equiv F(b_1, \ldots, b_m)\ (\sigma_1).$$

From the relations in the second column, we obtain

$$F(b_1, \ldots, b_m) \equiv F(a_1', \ldots, a_m')\ (\sigma_2),$$

and so

$$F(a_1, \ldots, a_m) \equiv F(a_1', \ldots, a_m')\ (\sigma),$$

as asserted.

Corollary 1. *If the composition $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ of congruences $\sigma_1, \ldots, \sigma_k$ is an equivalence, then σ is also a congruence. The composition of two congruences on an algebraic system $\mathfrak{A}$ is a congruence on $\mathfrak{A}$ if and only if their composition is commutative.*

For, by Section 1.4, the composition $\sigma_1\sigma_2$ is an equivalence if and only if $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

No matter what binary relations σ_α are defined in an algebraic system $\mathfrak{A}$, there is always a congruence on $\mathfrak{A}$ containing all σ_α. An

example of one is the universal congruence A^2. By Theorem 4, the intersection σ of all congruences on $\mathfrak{A}$ containing all σ_α is a congruence on $\mathfrak{A}$. It is the smallest congruence containing the relations σ_α.

The construction of the smallest equivalence σ containing a family of equivalences $\sigma_\alpha (\alpha \in \Sigma)$ was described in Section 1.4. It turned out that σ is the union of all possible compositions of equivalences in the family.

Theorem 6. *The smallest congruence σ containing a given family of congruences $\sigma_\alpha (\alpha \in \Sigma)$ on an algebraic system $\mathfrak{A}$ is the union of all possible compositions of congruences of the form $\sigma_{\alpha_1}\sigma_{\alpha_2}\dots\sigma_{\alpha_k}$. Thus the smallest congruence containing the family σ_α coincides with the smallest equivalence containing these congruences.*

We merely have to show that the union σ of compositions of the form $\sigma_{\alpha_1}\dots\sigma_{\alpha_k}$ $(k = 2, 3, \dots)$ is a congruence. Let F be a basic operation and let $a_1, a_1', \dots, a_m, a_m'$ be elements in $\mathfrak{A}$ for which relations (1) are true. This means that for suitable compositions $\varrho_1, \dots, \varrho_m$ the following relations are true:

$$a_1 \equiv a_1'(\varrho_1),$$
$$\dots\dots\dots$$
$$a_m \equiv a_m'(\varrho_m).$$

Since $\varrho_1, \dots, \varrho_m$ are reflexive, we have $\varrho_i \subseteq \varrho_1\varrho_2\dots\varrho_m$ and hence

$$a_1 \equiv a_1'(\varrho_1\varrho_2\dots\varrho_m),$$
$$\dots\dots\dots$$
$$a_m \equiv a_m'(\varrho_1\varrho_2\dots\varrho_m).$$

By Theorem 4, the composition $\varrho_1\varrho_2\dots\varrho_m$ is a stable relation and so

$$F(a_1, \dots, a_m) \equiv F(a_1', \dots, a_m')\ (\varrho_1\varrho_2\dots\varrho_m).$$

But $\varrho_1\dots\varrho_m \subseteq \sigma$ and hence we obtain (2). Theorem 6 is proved.

It is clear that if the given relations σ_α are equivalences rather than congruences, then the smallest congruence containing all σ_α will merely contain the smallest equivalence containing all σ_α.

In connection with the remarks of Section 1.4, we can make a further statement.

Corollary 2. *If σ_1 and σ_2 are commuting congruences, then their composition is the smallest congruence containing σ_1 and σ_2.*

For, the smallest congruence σ containing σ_1 and σ_2 also contains $\sigma_1\sigma_2$. On the other hand, by Corollary 1, $\sigma_1\sigma_2$ is a congruence containing σ_1 and σ_2. Hence $\sigma = \sigma_1\sigma_2$.

2.5 Direct Products. We have already studied cartesian products of a finite number of sets. We shall now extend the notion to any family of sets. In so doing, we shall introduce the direct (cartesian) product of an arbitrary family of equitype algebraic systems.

Suppose that to each element α of a nonempty index set A there corresponds an algebraic system

$$\mathfrak{M}_\alpha = \langle M_\alpha, \{F_\xi^{(\alpha)}\}, \{P_\eta^{(\alpha)}\}\rangle \qquad (\xi < \mu,\ \eta < \nu) \tag{1}$$

of given fixed type $\tau = \langle m_0, \ldots, m_\xi, \ldots; n_0, \ldots, n_\eta, \ldots\rangle$ $(\xi < \mu, \eta < \nu)$. The *cartesian* or *direct product* of the family of sets M_α $(\alpha \in A)$ is the totality of functions f from A into the union $\bigcup M_\alpha$ satisfying

$$f(\alpha) \in M_\alpha \qquad (\alpha \in A). \tag{2}$$

The cartesian product of family M_α will be denoted by any one of the expressions

$$\prod_{\alpha \in A} M_\alpha, \quad \prod M_\alpha \quad (\alpha \in A), \quad \prod M_\alpha,$$

Condition (2) says that element f of the cartesian product is specified by choosing element $f(\alpha) = f_\alpha$ in each set M_α. Conversely, each such combined "choice" of elements f_α in the sets M_α uniquely determines a function f satisfying (2) and hence determines an element in the cartesian product.

Suppose that each M_α is nonempty. Is then the cartesian product $\prod M_\alpha$ necessarily nonempty? If the index set A is finite, the answer is undoubtedly in the affirmative. The fact that each M_α is nonempty means that at least one element can be "chosen" from M_α. If such a selection is made for each α, the combined choice yields an element of the cartesian product. If A is an infinite set, one has to make infinitely many choices. The possibility of choosing a finite or infinite number of objects is taken as a postulate in set theory. It is called *Zermelo's Axiom* or the *Axiom of Choice*. We shall state it in the following form.

Axiom of Choice. *The cartesian product of any indexed family of nonempty sets is a nonempty set.*

Quite frequently the index set is the set N of natural numbers. In this case, a function f is well-defined if we prescribe the sequence of values

$$f(0), f(1), \ldots, f(k), \ldots \tag{3}$$

and f may therefore be identified with the sequence.

Until now we have been discussing the cartesian product $D = \prod M_\alpha$ of the carriers M_α of algebraic systems $\mathfrak{M}_\alpha$. We now shall define operations

F_ξ and predicates P_η on D and thereby convert D into an algebraic system $\mathfrak{D} = \langle D, \{F_\xi\}, \{P_\eta\}\rangle$ $(\xi < \mu,\ \eta < \nu)$ of the same type as that of each system $\mathfrak{M}_\alpha$.

Let $f_1, \ldots, f_{n_\eta}$ be elements of D. By definition, we say

$$P_\eta(f_1, \ldots, f_{n_\eta}) = \mathsf{T}$$

if and only if for each $\alpha \in A$,

$$P_\eta^{(\alpha)}\big(f_1(\alpha), \ldots, f_{n_\eta}(\alpha)\big) = \mathsf{T}.$$

Similarly, the value of $F_\xi(f_1, \ldots, f_{m_\xi})$ is the element of $f \in D$ defined by the conditions

$$f(\alpha) = F_\xi^{(\alpha)}\big(f_1(\alpha), \ldots, f_{m_\xi}(\alpha)\big) \quad (\alpha \in A).$$

The algebraic system $\mathfrak{D} = \langle D, \{F_\xi\}, \{P_\eta\}\rangle$ so constructed is called the *direct product* of the indexed family of systems M_α. It is denoted as in the case of a cartesian product of sets by

$$\prod_{\alpha \in A} \mathfrak{M}_\alpha, \quad \prod \mathfrak{M}_\alpha \quad (\alpha \in A), \quad \prod \mathfrak{M}_\alpha.$$

If the index set A comprises the numbers $1, 2, \ldots, k$, the direct product $\prod \mathfrak{M}_\alpha$ is often expressed as $\mathfrak{M}_1 \times \cdots \times \mathfrak{M}_k$ and its elements are identified with the sequences $(c_1, \ldots, c_k)$ $(c_i \in M_i)$. Similar notation is also used when the index set is the set of natural numbers. If the $\mathfrak{M}_\alpha$ are all the same system $\mathfrak{M}$, then the product is called the *direct power of* $\mathfrak{M}$.

Let us clarify the above definitions by a few examples. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the same system $\langle N, +, \leqq\rangle$. Consider their product $\mathfrak{D} = \mathfrak{M}_1 \times \mathfrak{M}_2$. The elements of the product are pairs (a, b) of natural numbers. The addition of pairs in $\mathfrak{D}$ is defined by

$$(a, b) + (c, d) = (a + c, b + d),$$

and the relation $\leqq$ by the condition

$$(a, b) \leqq (c, d) \Leftrightarrow a \leqq c \text{ and } b \leqq d.$$

In particular, we have

$$(1, 3) \leqq (1, 4), \quad (1, 3) \leqq (2, 5),$$

But

$$(1, 3) \not\leqq (2, 1), \quad (2, 1) \not\leqq (1, 3).$$

Thus $\mathfrak{D}$ is partially ordered.

Similarly, let $\mathfrak{M}_i$ $(i = 1, 2, 3, \ldots)$ represent the same model $\langle N, E\rangle$, E being the predicate "is an even number". The elements of the direct product $\mathfrak{D} = \prod \mathfrak{M}_i$ are infinite sequences $a = (a_1, a_2, \ldots, a_k, \ldots)$ of natural numbers such that $E(a) = \mathsf{T}$, i.e., each sequence is "even" if and only if all of its terms are even.

Consider again an arbitrary direct product $\mathfrak{D} = \prod \mathfrak{M}_\alpha$ of algebraic systems $\mathfrak{M}_\alpha$ $(\alpha \in A)$. For each fixed $\alpha \in A$, π_α: $f \to f(\alpha)$ $(f \in D)$ is a

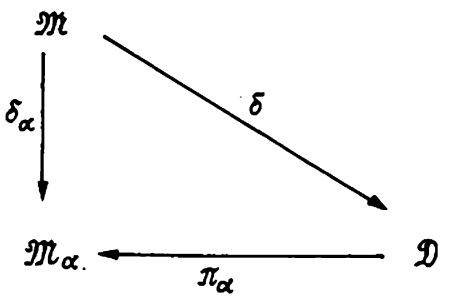

Fig. 1

mapping of D onto M_α. From the way direct product was defined it is immediately apparent that π_α is a homomorphism of $\mathfrak{D}$ onto $\mathfrak{M}_\alpha$, called the *projection of* $\mathfrak{D}$ *on* $\mathfrak{M}_\alpha$.

A family of homomorphisms δ_α $(\alpha \in A)$ of an algebraic system $\mathfrak{M}$ into the algebraic systems $\mathfrak{M}_\alpha$ is said to be *complete* if it is possible to separate any two distinct elements of $\mathfrak{M}$ by means of these homomorphisms. In other words, for any a and b in $\mathfrak{M}$, $a = b$ whenever $a\delta_\alpha = b\delta_\alpha$ for each α.

Theorem 1. *Let $\alpha \to \mathfrak{M}_\alpha$ be a mapping of the index set A onto a family $\{\mathfrak{M}_\alpha\}$ of equitype algebraic systems. The direct product $\mathfrak{D} = \prod \mathfrak{M}_\alpha$ and projections π_α: $\mathfrak{D} \to \mathfrak{M}_\alpha$ have the following properties:*

a) *The family $\{\pi_\alpha\}$ is complete;*

b) *If $\mathfrak{M}$ is an arbitrary algebraic system of the same type as $\mathfrak{M}_\alpha$ on which there are defined homomorphisms δ_α into $\mathfrak{M}_\alpha$, then there exists a homomorphism δ of $\mathfrak{M}$ into $\mathfrak{D}$ such that $\delta_\alpha = \delta\pi_\alpha$* (Fig. 1).*)

The statement a) is trivial. Indeed, let a and $b \in D$ and let $a\pi_\alpha = b\pi_\alpha$, $\alpha \in A$. Then $a(\alpha) = b(\alpha)$ for each $\alpha \in A$ and hence $a = b$.

*) In this case, the diagram in Fig. 1 is said to be *commutative* [*Trans.*].

Let us prove statement b). We define a function $m\delta$ on A corresponding to any $m \in \mathfrak{M}$ by means of

$$(m\delta)(\alpha) = m\delta_\alpha \quad (\alpha \in A).$$

Since $m\delta_\alpha \in \mathfrak{M}_\alpha$, the mapping $m \to m\delta$ is a mapping of $\mathfrak{M}$ into $\mathfrak{D}$. It clearly satisfies the requirement $\delta_\alpha = \delta\pi_\alpha$. It remains to show that δ is a homomorphism. Let P be any basic predicate of arity n and let $P(a_1, \ldots, a_n) = \mathsf{T}$ for some $a_1, \ldots, a_n$ in $\mathfrak{M}$. Then $P(a_1\delta_\alpha, \ldots, a_n\delta_\alpha) = \mathsf{T}$ for each $\alpha \in A$ and so by virtue of the definition of direct product, $P(a_1\delta, \ldots, a_n\delta) = \mathsf{T}$. The truth of the homomorphism conditions concerning basic operations can be established in a similar way.

An important fact is that Theorem 1 has the following converse.

Theorem 2. *Let $\alpha \to \mathfrak{M}_\alpha$ be a mapping of index set A onto a family $\{\mathfrak{M}_\alpha\}$ of equitype algebraic systems and let $\varkappa_\alpha$ be a given family of homomorphisms of an algebraic system $\mathfrak{N}$ onto the $\mathfrak{M}_\alpha$ satisfying*

a') *The family of homomorphisms $\varkappa_\alpha$ $(\alpha \in A)$ is complete;*

b') *If $\mathfrak{M}$ is an arbitrary algebraic system of the same type as $\mathfrak{M}_\alpha$ on which there are defined homomorphisms δ_α into $\mathfrak{M}_\alpha$, then there exists a homomorphism δ of $\mathfrak{M}$ into $\mathfrak{N}$ for which $\delta_\alpha = \delta\varkappa_\alpha$ for all $\alpha \in A$* (cf. Fig. 1).

Then there exists an isomorphism γ of $\mathfrak{N}$ onto $\mathfrak{D} = \prod \mathfrak{M}_\alpha$ under which the homomorphisms $\varkappa_\alpha$ are carried into the projections π_α, i.e., *such that $\varkappa_\alpha = \gamma\pi_\alpha$ $(\alpha \in A)$.*

To this end, we take the system $\mathfrak{M}$ and homomorphisms δ_α to be the direct product $\mathfrak{D}$ and projections π_α. According to b'), there exists a homomorphism δ of $\mathfrak{D}$ into $\mathfrak{N}$ such that

$$\pi_\alpha = \delta\varkappa_\alpha \quad (\alpha \in A). \tag{4}$$

On the other hand, by Theorem 1, there exists a homomorphism γ of $\mathfrak{N}$ into $\mathfrak{D}$ for which

$$\varkappa_\alpha = \gamma\pi_\alpha. \tag{5}$$

Comparing (4) and (5), we obtain

$$\pi_\alpha = \delta\gamma\pi_\alpha, \quad \varkappa_\alpha = \gamma\delta\varkappa_\alpha.$$

By the completeness of the families π_α and $\varkappa_\alpha$, this implies

$$\delta\gamma = \varepsilon_1, \quad \gamma\delta = \varepsilon_2, \tag{6}$$

where ε_1 and ε_2 are the respective identity mappings of $\mathfrak{D}$ and $\mathfrak{N}$ onto themselves. From (6) it follows that δ and γ are isomorphisms (see

Section 2.2). Thus the mapping γ is an isomorphism of $\mathfrak{N}$ onto $\mathfrak{D}$ satisfying (5), as asserted.

According to Theorem 2, conditions a) and b) characterize a direct product up to isomorphism. Hence, these conditions could serve as a second definition of direct product. The first definition provides an explicit construction for the product. The second definition merely describes its properties and it may therefore be called an axiomatic definition. Under the axiomatic definition, it is necessary to give a separate proof of the existence of a direct product. This is the shortcoming of the axiomatic definition. Its advantage is that it is formulated only in terms of homomorphic mappings and makes no mention of the elements of the algebraic systems involved. True, there are indirect references to the system elements in Theorem 2 since the completeness of the family of homomorphisms π_α is defined with the help of the elements of $\mathfrak{D}$. But these references may be eliminated if we make use of the following definition.

A family of homomorphisms δ_α $(\alpha \in A)$ of an algebraic system $\mathfrak{M}$ into system $\mathfrak{M}_\alpha$ is *quasicomplete* if for each pair of endomorphisms γ_1 and γ_2 of $\mathfrak{M}$ (*homomorphisms of* $\mathfrak{M}$ *into itself*), the fulfillment of

$$\gamma_1\delta_\alpha = \gamma_2\delta_\alpha \qquad (\alpha \in A)$$

implies $\gamma_1 = \gamma_2$.

From the way Theorem 2 was proved, it is apparent that condition a') may be replaced by the requirement that the family of homomorphisms κ_α $(\alpha \in A)$ be quasicomplete. It is also clear that every complete family of homomorphisms is quasicomplete.

Concepts that may be defined making use of only the notion of homomorphism are called *category concepts*. Thus Theorem 2 may be said to provide a category characterization of the direct product.

An immediate consequence of Theorem 2 is that direct products of respectively isomorphic algebraic systems are isomorphic. However we shall state this more precisely in the form of a general assertion.

Theorem 3. *Consider the products* $\mathfrak{D} = \prod_{\alpha\in A} \mathfrak{M}_\alpha$ *and* $\mathfrak{D}^* = \prod_{\beta\in B} \mathfrak{M}_\beta{}^*$ *of equitype algebraic systems. Suppose there exists a one-to-one mapping* φ *of* A *onto* B *and isomorphisms* ψ_α *of* $\mathfrak{M}_\alpha$ *onto* $\mathfrak{M}^*_{\alpha\varphi}$ $(\alpha \in A)$. *Then there exists an isomorphism* γ *of* $\mathfrak{D}$ *onto* $\mathfrak{D}^*$ *satisfying* $\pi_\alpha\psi_\alpha = \gamma\pi^*_{\alpha\varphi}$ $(\alpha \in A)$.

Consider the homomorphisms $\kappa_\beta = \pi_{\beta\varphi^{-1}}\psi_{\beta\varphi^{-1}}$ of $\mathfrak{D}$ onto $\mathfrak{M}_\beta{}^*$ $(\beta \in B)$. These homomorphisms satisfy conditions a') and b') of Theorem 2. Thus we can find an isomorphism γ of $\mathfrak{D}$ onto $\mathfrak{D}^*$ such that

$$\pi_{\beta\varphi^{-1}}\psi_{\beta\varphi^{-1}} = \gamma\pi_\beta{}^*.$$

From this we conclude that $\pi_\alpha\psi_\alpha = \gamma\pi^*_{\alpha\varphi}$.

Theorem 3 shows that up to isomorphism a direct product is independent of the order in which its factors are indexed by the elements of A. Because of this, Theorem 3 is sometimes called the commutative law for direct products. The customary associative law also carries over to direct products and can be expressed in the following general form.

Theorem 4. *Let A_γ $(\gamma \in C)$ be a partition of the index set A for $\mathfrak{D} = \prod_{\alpha \in A} \mathfrak{M}_\alpha$ into pairwise disjoint subsets. Then $\mathfrak{D}^* = \prod_{\gamma \in C} \left(\prod_{\alpha \in A_\gamma} \mathfrak{M}_\alpha \right)$ is isomorphic to $\mathfrak{D} = \prod_{\alpha \in A} \mathfrak{M}_\alpha$.*

Introduce the notation: $\mathfrak{D}_\gamma = \prod_{\alpha \in A_\gamma} \mathfrak{M}_\alpha$, π_α^γ is projection of $\mathfrak{D}_\gamma$ on $\mathfrak{M}_\alpha$, π_γ^* is projection of $\mathfrak{D}^*$ on $\mathfrak{D}_\gamma$ and π_α is projection of $\mathfrak{D}$ on $\mathfrak{M}_\alpha$.

Consider the homomorphisms $\varkappa_\alpha = \pi_\gamma^* \pi_\alpha^\gamma$ $(\alpha \in A_\gamma)$ of $\mathfrak{D}^*$ onto $\mathfrak{M}_\alpha$. Let us show that $\varkappa_\alpha$ and $\mathfrak{D}^*$ satisfy the hypotheses of Theorem 2. The quasicompleteness of the family of homomorphisms $\varkappa_\alpha$ is trivial since if γ_1 and γ_2 are endomorphisms, then the equations $\gamma_1 \varkappa_\alpha = \gamma_2 \varkappa_\alpha$ $(\alpha \in A_\gamma)$ lead to the relations

$$(\gamma_1 \pi_\gamma^*) \pi_\alpha^\gamma = (\gamma_2 \pi_\gamma^*) \pi_\alpha^\gamma \qquad (\alpha \in A_\gamma).$$

But the family of homomorphisms π_α^γ is complete for $\mathfrak{D}$ and so $\gamma_1 \pi_\gamma^* = \gamma_2 \pi_\gamma^*$ $(\gamma \in C)$. In turn, the family of homomorphisms π_γ^* is complete and so $\gamma_1 = \gamma_2$.

We now verify condition b'). Let the δ_α be given homomorphisms of an arbitrary system $\mathfrak{M}$ into the systems $\mathfrak{M}_\alpha$ $(\alpha \in A)$. By Theorem 1, to each $\gamma \in C$, there is a homomorphism ϱ_γ of $\mathfrak{M}$ into $\mathfrak{D}_\gamma$ for which

$$\delta_\alpha = \varrho_\gamma \pi_\alpha^\gamma \qquad (\alpha \in A_\gamma). \tag{7}$$

By the same Theorem 1, there is a homomorphism ϱ of $\mathfrak{M}$ into the product $\mathfrak{D}^* = \prod \mathfrak{D}_\gamma$ satisfying

$$\varrho_\gamma = \varrho \pi_\gamma^* \qquad (\gamma \in C). \tag{8}$$

From (7) and (8) we obtain

$$\delta_\alpha = \varrho \pi_\gamma^* \pi_\alpha^\gamma = \varrho \varkappa_\alpha \qquad (\alpha \in A),$$

and property b') is proved. From properties a') and b') of family $\{\varkappa_\alpha\}$, it follows by Theorem 2 that the required isomorphism σ of $\mathfrak{D}^*$ onto $\mathfrak{D}$ exists satisfying moreover

$$\pi_\alpha^* = \sigma \pi_\alpha \qquad (\alpha \in A).$$

An algebraic system $\mathfrak{M}$ is said to be *decomposable* into a direct product of systems $\mathfrak{M}_\alpha$ $(\alpha \in A)$ if $\mathfrak{M}$ is isomorphic to the direct product

$\prod \mathfrak{M}_\alpha$. The question of uniqueness of such a decomposition requires more detailed discussion. We shall confine ourselves here to stating one rather obvious decomposability criterion which nevertheless turns out to be useful in many cases.

Consider a direct product

$$\mathfrak{D} = \prod \mathfrak{M}_\alpha \quad (\alpha \in A)$$

of algebraic systems $\mathfrak{M}_\alpha$. Let π_α be the projection of $\mathfrak{D}$ on $\mathfrak{M}_\alpha$. The homomorphism π_α induces a kernel congruence on $\mathfrak{D}$ (see Section 2.4) which we shall denote by ϱ_α. If the systems $\mathfrak{M}_\alpha$ are algebras, we have

$$\mathfrak{M}_\alpha \cong \mathfrak{D}/\varrho_\alpha, \tag{9}$$

where $\cong$ symbolizes isomorphism. But if the $\mathfrak{M}_\alpha$ are not algebras, the isomorphism (9) need not hold. Consider, for example, the direct product of the one-element models $\langle\{1\}, E\rangle$ and $\langle\{2\}, E\rangle$, where $E(x)$ is the unary predicate "x is an even number". The product consists of a single pair $(1, 2)$ and $E(1, 2) = \mathsf{F}$. The two projections π_1 and π_2 are homomorphisms. But the quotient model $\mathfrak{D}/\varrho_2$ is the class $\{(1, 2)\}$ and $E(1, 2) = \mathsf{F}$ in it. Therefore the models $\langle\{2\}, E\rangle$ and $\mathfrak{D}/\varrho_2$ are not isomorphic.

An arbitrary family of congruences σ_α $(\alpha \in A)$ on an algebraic system $\mathfrak{M} = \langle M, \Omega\rangle$ will be called *complete* if

1) to each pair of distinct elements a and b of M, there exists a congruence σ_α in the family $\{\sigma_\alpha\}$ separating a and b, i.e., $a \not\equiv b\,(\sigma_\alpha)$ and

2) whenever $P(a_1, \ldots, a_n) = \mathsf{F}$ for some basic predicate P and $a_1, \ldots, a_n$ in M, there exists a congruence σ_α such that

$$x_1 \equiv a_1\,(\sigma_\alpha), \ldots, x_n \equiv a_n\,(\sigma_\alpha) \Rightarrow P(x_1, \ldots, x_n) = \mathsf{F}$$

for any $x_1, \ldots, x_n$ in M.

From the completeness of the family of projections π_α of the direct product $\prod \mathfrak{M}_\alpha$ on its factors it follows that the family of congruences ϱ_α necessarily satisfies condition 1). It is easy to see that the family also satisfies condition 2). For, if $P(a_1, \ldots, a_n) = \mathsf{F}$ for some basic predicate P and elements $a_1, \ldots, a_n$ of the product, then by definition of product,

$$P(a_1\pi_\alpha, \ldots, a_n\pi_\alpha) = \mathsf{F}$$

for a suitable α.

Suppose $x_i \equiv a_i\,(\varrho_\alpha)$ $(i = 1, \ldots, n)$ and hence $x_i\pi_\alpha = a_i\pi_\alpha$. If $P(x_1, \ldots, x_n)$ were true, we would have $P(x_1\pi_\alpha, \ldots, x_n\pi_\alpha) = \mathsf{T}$ (the mapping π_α is a homomorphism) and hence $P(a_1\pi_\alpha, \ldots, a_n\pi_\alpha) = \mathsf{T}$ contrary to assumption. Thus, the congruences ϱ_α satisfy 2).

A family of congruences σ_α on an algebraic system $\mathfrak{M}$ will be called *independent* if to any mapping $\varphi\colon \alpha \to a_\alpha$ of the index set A into $\mathfrak{M}$ there is an element $a \in M$ such that

$$a \equiv a_\alpha(\sigma_\alpha) \quad \text{for all } \alpha \in A.$$

It is immediately apparent from the way $\mathfrak{D} = \prod \mathfrak{M}_\alpha$ is defined that a family of kernel congruences ϱ_α on $\mathfrak{D}$ is independent.

Thus if an algebraic system $\mathfrak{M}$ has been decomposed into a direct product of systems $\mathfrak{M}_\alpha$ $(\alpha \in A)$, there exists a complete independent family of congruences σ_α $(\alpha \in A)$ on $\mathfrak{M}$ such that $\mathfrak{M}_\alpha$ is the one-to-one homomorphic image of $\mathfrak{M}/\sigma_\alpha$.

This assertion admits the following converse.

Theorem 5. *If there exists a complete independent family of congruences σ_α $(\alpha \in A)$ on an algebraic system $\mathfrak{M}$, then $\mathfrak{M}$ is isomorphic to the direct product of the quotient systems $\mathfrak{M}/\sigma_\alpha$ $(\alpha \in A)$.*

Let $a\sigma_\alpha$ denote the σ_α-equivalence class of $\mathfrak{M}$ that contains element a. To each $a \in \mathfrak{M}$, define a function f_a on A by setting $f_a(\alpha) = a\sigma_\alpha$ and consider the mapping $\varphi\colon a \to f_a$ of $\mathfrak{M}$ into the direct product $\mathfrak{D} = \prod \mathfrak{M}/\sigma_\alpha$. The fact that the congruences σ_α are independent implies that φ is a homomorphism of $\mathfrak{M}$ *onto* $\mathfrak{D}$. Condition 1) in the definition of completeness shows that φ is a one-to-one mapping. Finally, condition 2) in the definition of completeness assures that φ is an isomorphism. Indeed, suppose $P(a_1, \ldots, a_n) = \mathsf{F}$ for some basic predicate P and elements $a_1, \ldots, a_n$ in $\mathfrak{M}$. By condition 2) there exists an index α such that $P(a_1\sigma_\alpha, \ldots, a_n\sigma_\alpha) = \mathsf{F}$ in $\mathfrak{M}/\sigma_\alpha$. But then $P(f_{a_1}, \ldots, f_{a_n}) = \mathsf{F}$ by the definition of direct product. The validity of the corresponding statement for each basic operation is proved in a similar way.

The following remark may be made apropos of Theorem 5 and the assertion preceding it. Suppose a model $\mathfrak{M}$ is isomorphic to a direct product $\mathfrak{M}_1 \times \mathfrak{M}_2$. Then there exist congruences ϱ_1 and ϱ_2 on $\mathfrak{M}$ corresponding to the projections π_1 and π_2 on $\mathfrak{M}_1$ and on $\mathfrak{M}_2$. We cannot say that $\mathfrak{M}_1$ is isomorphic to $\mathfrak{M}/\varrho_1$ or that $\mathfrak{M}_2$ is isomorphic to $\mathfrak{M}/\varrho_2$. But by Theorem 5, we can say that $\mathfrak{M}$ is isomorphic to $\mathfrak{M}/\varrho_1 \times \mathfrak{M}/\varrho_2$. These subtle points disappear when algebras are considered. For in that case, condition 2) in the definition of completeness does not occur and so the mappings of $\mathfrak{M}/\varrho_\alpha$ onto $\mathfrak{M}_\alpha$ are isomorphisms.

We state one further consequence of Theorem 5. The *index* of an equivalence relation θ on a set M is the number (cardinality of the set) of equivalence classes in M/θ. From Theorem 5 and the result preceding it, we can conclude that an algebraic system $\mathfrak{M}$ is decomposable into a direct product of finite systems if and only if there exists a complete independent family of congruences on $\mathfrak{M}$ of finite index.

Closely related to the concept of direct product of algebraic systems is the notion of finite approximability.

An algebraic system $\mathfrak{M} = \langle M, \Omega \rangle$ is said to be *finitely approximable* if it has a complete family of congruences of finite index, or in other words, if to each predicate P in $\mathfrak{M}$ (either basic or coinciding with the equality relation $(=)$) and to each n elements $a_1, \ldots, a_n$ in M $(n = n(P))$ for which $P(a_1, \ldots, a_n) = \mathsf{F}$, there exists a homomorphism φ of $\mathfrak{M}$ into a finite system $\mathfrak{N}$ such that $P(a_1\varphi, \ldots, a_n\varphi) = \mathsf{F}$.

From this is follows that *a subsystem $\mathfrak{M}_1$ of a finitely approximable system $\mathfrak{M}$ is finitely approximable.*

Theorem 6. *An algebraic system $\mathfrak{M} = \langle M, \Omega \rangle$ is finitely approximable if and only if it is isomorphically embeddable in a direct product of finite systems.*

For, if $\mathfrak{M} = \prod \mathfrak{M}_\alpha$ $(\alpha \in A)$ and all $\mathfrak{M}_\alpha$ are finite, then the kernel congruences corresponding to the projections $\pi_\alpha\colon \mathfrak{M} \to \mathfrak{M}_\alpha$ have finite indices and comprise a complete family.

Conversely, suppose $\mathfrak{M}$ is a finitely approximable system. Consider all possible finite homomorphic images $\mathfrak{M}_\alpha$ $(\alpha \in A)$ of $\mathfrak{M}$ and suppose $\varphi_\alpha\colon \mathfrak{M} \to \mathfrak{M}_\alpha$ are fixed homomorphisms. Let $\mathfrak{D} = \prod \mathfrak{M}_\alpha$ $(\alpha \in A)$. By Theorem 1, there exists a homomorphism $\varphi\colon \mathfrak{M} \to \mathfrak{D}$ such that $\varphi_\alpha = \varphi\pi_\alpha$. If a and b belong to M, $a \neq b$, then by hypothesis there is a homomorphism φ_α such that $a\varphi_\alpha \neq b\varphi_\alpha$, i. e., $a\varphi\pi_\alpha \neq b\varphi\pi_\alpha$. Hence, $a\varphi \neq b\varphi$. Similarly, if $P \in \Omega_P$, and if $P(a_1\varphi, \ldots, a_n\varphi) = \mathsf{T}$ for some $a_1, \ldots, a_n$ in M, then

$$P(a_1\varphi\pi_\alpha, \ldots, a_n\varphi\pi_\alpha) = P(a_1\varphi_\alpha, \ldots, a_n\varphi_\alpha) = \mathsf{T}$$

for all $\alpha \in A$. This implies $P(a_1, \ldots, a_n) = \mathsf{T}$. Thus φ is an isomorphism of $\mathfrak{M}$ into $\mathfrak{D}$.

A product of the form $\mathfrak{D}_B = \prod \mathfrak{M}_\beta$ $(\beta \in B)$, where B is a proper subset of A, is a *subproduct* of direct product $\mathfrak{D}_A = \prod \mathfrak{M}_\alpha$ $(\alpha \in A)$. Let $C = A - B$. Then the direct product $\mathfrak{D}_C = \prod \mathfrak{M}_\gamma$ $(\gamma \in C)$ is the *complementary subproduct* to $\mathfrak{D}_B$. Each element $f \in \mathfrak{D}_A$ is a function defined on A with values in the respective factors of $\mathfrak{D}_A$. Let f_B and f_C be the restrictions of f to B and C, respectively. The mappings π_B: $f \to f_B$ and $\pi_C\colon f \to f_C$ are clearly homomorphisms of $\mathfrak{D}_A$ onto the subproducts $\mathfrak{D}_B$ and $\mathfrak{D}_C$ while the mapping $f \to (f_B, f_C)$ is an isomorphism of $\mathfrak{D}_A$ onto $\mathfrak{D}_B \times \mathfrak{D}_C$ (see Theorem 4). The mapping π_B is called the *projection* of $\mathfrak{D}_A$ on $\mathfrak{D}_B$.

Let us see whether the projection π_B $(B \neq A)$ is an isomorphism. If the complementary product $\mathfrak{D}_C$ contains two distinct elements u and v, then the pairs (f_B, u) and (f_B, v) have the same projection f_B on $\mathfrak{D}_B$ and so the projection π_B is not one-to-one. Thus a necessary condition for π_B

to be an isomorphism is that each of the factors $\mathfrak{M}_\gamma$ ($\gamma \in C$) be a one-element algebraic system.

It should be noted that there are just finitely many pairwise non-isomorphic one-element algebraic systems of fixed finite type and, up to isomorphism, just a single one-element algebra of each finite or infinite type.

A direct product of algebraic systems each isomorphic to a fixed one-element system is isomorphic to that system. Therefore, if a direct product $\mathfrak{D} = \prod \mathfrak{M}_\alpha$ contains isomorphic one-element factors, Theorem 4 says that all of them may be deleted except for one. The reduced product will be isomorphic to the original one.

An algebraic system is called *trivial* if it has a single element and all its basic predicates have the value T. Since an algebra has no basic predicates, a trivial algebra is the same as a one-element algebra. For models, these two notions do not coincide. For example, $\langle\{1\}; E\rangle$ (E is the predicate "is an even number") is a one-element model but it is not trivial since $E(1) = \mathsf{F}$.

It is easy to see that trivial factors may be deleted from a direct product containing at least one nontrivial factor. The value of the product remains the same up to isomorphism. In general, nontrivial one-element factors cannot be deleted.

A *proper* direct product is a product having more than one factor and for which none of its projections on proper subproducts is an isomorphism. For example, the direct product of two or more factors each containing more than one element is necessarily proper.

An algebraic system is *directly indecomposable* if it is not isomorphic to any direct product of two or more systems each having more than one element. An algebraic system is *absolutely directly indecomposable* if it is isomorphic to no proper direct product. In view of the above, these two ideas coincide for algebras.

To illustrate the ideas just introduced we prove the following statement.

Theorem 7. *Each finite algebraic system $\mathfrak{M}$ of finite type is decomposable into a direct product of a finite number of absolutely directly indecomposable systems. A one-element algebraic system of finite type is absolutely directly indecomposable if and only if it has at most one false basic predicate.*

Let us prove the first assertion. If the given system is absolutely indecomposable, then $\mathfrak{M} = \mathfrak{M}$ is the required decomposition. If $\mathfrak{M}$ is decomposable, then $\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2$, where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are nontrivial factors. If both factors are absolutely indecomposable, then everything is proved. If one of them can be decomposed, then upon doing it, we obtain a longer decomposition of $\mathfrak{M}$ and so on. We must show that this

process of decomposing factors terminates. This is obvious for algebras since a proper product of algebra contains no one-element factors and so each decomposition results in factors with a fewer number of elements. But if the system in question is not an algebra, i.e., the signature of $\mathfrak{M}$ contains predicate symbols, this necessitates a little further discussion.

A *sequent* of an algebraic system $\mathfrak{M} = (M; F_1, \ldots, F_s; P_1, \ldots, P_t)$ is a sequence of the form $(P_i, a_1, \ldots, a_{n_i})$, where $a_1, \ldots, a_{n_i}$ are elements of M. A sequent $(P_i, a_i, \ldots, a_{n_i})$ is said to be positive if $P_i(a_i, \ldots, a_{n_i}) = \mathsf{T}$ in $\mathfrak{M}$. The height of a system $\mathfrak{M}$ is defined as the pair $(u, t - v)$, where u is the order of $\mathfrak{M}$ and v is the number of positive sequents which $\mathfrak{M}$ has. Since $\mathfrak{M}$ is finite and of finite type, the numbers u and v are finite. A trivial system has height $(1, 0)$. We order the various heights lexicographically setting

$$(u, v) \leqq (x, y) \Leftrightarrow u < x \vee (u = x \,\&\, v \leqq y),$$

wherein the symbol $\vee$ stands for the conjunction "or". It is clear that if a finite system $\mathfrak{M}$ can be mapped homomorphically on a system $\mathfrak{M}_1$, then the height of $\mathfrak{M}$ is no less than the height of $\mathfrak{M}_1$ and the two systems have equal height only when the homomorphism is an isomorphism.

Thus, when the factors in proper direct products are decomposed, the heights of the new factors are less than the heights of the factors being decomposed. Hence it is apparent that the process of step-by-step decomposition of factors cannot be continued indefinitely, as asserted. The height of a one-element algebraic system of finite type having all true basic predicates but for one is $(1, 1)$. If decomposed into a direct product, there must occur factors of height $(1, 0)$ and this is impossible in a proper product. Therefore the specified one-element systems are absolutely directly indecomposable.

If in the system $\mathfrak{M} = \langle |a|; F_1, \ldots, F_s; P_1, \ldots, P_t \rangle$ we have

$$P_i(a, \ldots, a) = \mathsf{F}, \qquad P_j(a, \ldots, a) = \mathsf{T}$$

$$(i = 1, \ldots, k; \qquad j = k + 1, \ldots, t),$$

then

$$\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2 \times \ldots \times \mathfrak{M}_k,$$

where $\mathfrak{M}_i$ is a one-element system for which all predicates except P_i are true.

2.6 Operations on Cardinal and Ordinal Numbers. Let $\mathfrak{a}$ and $\mathfrak{b}$ be arbitrary cardinal numbers and let $\mathfrak{a} = |A|$, $\mathfrak{b} = |B|$ and $A \cap B = \varnothing$.

By definition, we set

$$\mathfrak{a} + \mathfrak{b} = |A \cup B|,$$
$$\mathfrak{a}\mathfrak{b} = |A \times B|,$$
$$\mathfrak{a}^{\mathfrak{b}} = |A^B|,$$

where A^B is the set of all mappings of B into A. It is easy to prove that the results of these operations depend just on $\mathfrak{a}$ and $\mathfrak{b}$ and not on the choice of the sets A and B. The use of the simplest properties of union and cartesian product leads to the following laws:

$$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}, \qquad \mathfrak{a}(\mathfrak{b}\mathfrak{c}) = (\mathfrak{a}\mathfrak{b})\mathfrak{c},$$
$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}, \qquad \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}.$$

It is also easy to prove that

$$\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}}\mathfrak{a}^{\mathfrak{c}}, \quad (\mathfrak{a}\mathfrak{b})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{c}}\mathfrak{b}^{\mathfrak{c}}, \quad (\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\mathfrak{c}}.$$

If we associate with each subset of A its characteristic function (see Section 2.1), we obtain a one-to-one mapping of $\mathscr{P}(A)$ onto the set 2^A, where 2 stands for the set $\{0, 1\}$. Therefore the cardinality of the set $\mathscr{P}(A)$ of all subsets of a set A is equal to $2^{|A|}$. In other words,

$$|\mathscr{P}(A)| = 2^{|A|}.$$

Applying Cantor's theorem, we obtain

$$2^{\mathfrak{a}} > \mathfrak{a}$$

for any cardinal number $\mathfrak{a}$.

The question of the existence of an intermediary cardinal between $\aleph_0$ and $2^{\aleph_0}$ is the continuum problem.*)

Let $\{\mathfrak{a}_i \mid i \in I\}$ be an arbitrary set of cardinals $\mathfrak{a}_i = |A_i|$ such that $A_i \cap A_j = \varnothing$ for $i \neq j$. By definition,

$$\sum_{i \in I} \mathfrak{a}_i = |\bigcup_{i \in I} A_i|, \qquad \prod_{i \in I} \mathfrak{a}_i = |\prod_{i \in I} A_i|.$$

Theorem 1. *If* $\mathfrak{a}_i = \mathfrak{a}$ *for all* $i \in I$, *then*

$$\sum_{i \in I} \mathfrak{a}_i = |I|\,\mathfrak{a}. \tag{1}$$

*) P. J. Cohen has shown that the continuum problem is undecidable within the framework of the Zermelo-Fraenkel set-theoretic axioms with the Axiom of Choice included (*The independence of the continuum hypothesis*, I. Proc. Nat. Acad. Sci. **50**, 1963; II, **51**, 1964) [*Trans.*].

For, if $\mathfrak{a} = |A|$, then $|I|\,\mathfrak{a} = |I \times A|$. To each $i \in I$, let (i, A) denote the set of pairs of the form (i, a) with $a \in A$. Then

$$\bigcup_{i \in I} (i, A) = I \times A. \tag{2}$$

Since $(i, A) \cap (j, A) = \varnothing$ for $i \neq j$, equation (1) follows immediately from (2) since $|(i, A)| = |A| = \mathfrak{a}$.

For example, taking I to be the set of natural numbers, we obtain

$$\mathfrak{a} + \mathfrak{a} + \mathfrak{a} + \ldots = \aleph_0 \mathfrak{a} \tag{3}$$

for any cardinal number $\mathfrak{a}$.

A few further easily proved identities for cardinal numbers are

$$\aleph_0 \aleph_0 = \aleph_0, \qquad \mathfrak{a} + \mathfrak{b} = \mathfrak{a}, \tag{4}$$

where $\mathfrak{a}$ is infinite and $\mathfrak{b}$ is finite or countable.

Indeed, the set of pairs (m, n) of natural numbers is easily enumerated. We merely have to assign the natural numbers in succession to the pairs for each value of the sum $m + n = 0, 1, \ldots$ If a set A is infinite, it contains a countable subset $B = \{a_1, a_2, \ldots\}$. If $C = A - B$ and $|D| = \mathfrak{b}$, then

$$A \cup D = C \cup (B \cup D), \qquad |A \cup D| = |C \cup B| = |A|.$$

The sum of two order types was defined in Section 1.7. The product $\alpha\beta$ of order types $\alpha = o(A)$ and $\beta = o(B)$ is defined as the order type of the cartesian product $B \times A$ linearly ordered lexicographically: $(a, b) < (c, d)$ if and only if $a < c$ or $a = c$ and $b < d$.

For example, $\omega 2$ is the order type of the set of pairs $\{(1, n), (2, n)\}$ $(n = 1, 2, \ldots)$ linearly ordered lexicographically as follows:

$$(1, 1) < (1, 2) < (1, 3) < \ldots$$
$$\ldots < (2, 1) < (2, 2) < (2, 3) < \ldots$$

Hence, $\omega 2 = \omega + \omega$. The product 2ω is the order type of the set of pairs $\{(n, 1), (n, 2)\}$ $(n = 1, 2, \ldots)$ arranged in the order

$$(1, 1) < (1, 2) < (2, 1) < (2, 2) < (3, 1) < (3, 2) < \ldots$$

Hence, $2\omega = \omega$.

The definition implies in particular that *the product $\alpha\beta$ of two ordinal numbers α and β is an ordinal number* (*as was* $\alpha + \beta$).

Theorem 2. *Let α, β, and ζ be ordinal numbers satisfying $\zeta < \alpha\beta$. Then there exist ordinal numbers ξ and η such that*

$$\zeta = \alpha\eta + \xi, \quad \xi < \alpha, \quad \eta < \beta.$$

To show this, let $\alpha = o(A)$ and $\beta = o(B)$. Then $\alpha\beta$ is the type of $B \times A$ well-ordered lexicographically. Since $\zeta < \alpha\beta$, ζ is the type of some initial segment $P_{(b,a)}$ consisting of those pairs strictly less than (b, a), i.e., the pairs (y, x) $(y \in B, x \in A)$ such that either $y < b$ or $y = b$ and $x < a$. Let P_a and P_b be initial segments of the well-ordered sets A and B determined by the respective elements a and b. If $\xi = o(P_a)$ and $\eta = o(P_b)$, then $\zeta = \alpha\eta + \xi$ since

$$P_{(b,a)} = P_b \times A \cup (b, P_a),$$

where (b, P_a) is the set of pairs of the form (b, x) with x in P_a.

It is easy to see that for given α, β and ζ, the numbers ξ and η are uniquely determined by the conditions of Theorem 2.

Corollary. *To any ordinals α and ζ with $\alpha > 0$, there exist ordinals ξ and η such that*

$$\zeta = \alpha\eta + \xi, \quad \xi < \alpha. \tag{5}$$

To prove this, we merely have to apply Theorem 2 with $\beta = \zeta + 1$.

Theorem 3. *Every infinite cardinal number $\mathfrak{a}$ satisfies*

$$\aleph_0\mathfrak{a} = \mathfrak{a}.$$

To this end, let α be an ordinal number of cardinality $\mathfrak{a}$. By (5), there are ordinal numbers η and n such that

$$\alpha = \omega\eta + n, \quad n < \omega.$$

Let $\mathfrak{b}$ denote the cardinality of η. Since n is finite, we find in view of (4) that

$$\mathfrak{a} = \aleph_0\mathfrak{b},$$

$$\aleph_0\mathfrak{a} = \aleph_0^2\mathfrak{b} = \aleph_0\mathfrak{b} = \mathfrak{a}.$$

Corollary 1. *If at least one of the cardinals $\mathfrak{a}$ or $\mathfrak{b}$ is infinite, then*

$$\mathfrak{a} + \mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b}).$$

Indeed, suppose for example that $\mathfrak{a} \geqq \mathfrak{b}$. Then

$$\mathfrak{a} \leqq \mathfrak{a} + \mathfrak{b} \leqq \mathfrak{a} + \mathfrak{a} = 2\mathfrak{a} \leqq \aleph_0 \mathfrak{a} = \mathfrak{a},$$

and so $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$.

Theorems 1 and 3 also imply

Corollary 2. *Every infinite cardinal number* $\mathfrak{a}$ *satisfies*

$$\mathfrak{a} + \mathfrak{a} + \mathfrak{a} + \ldots = \mathfrak{a},$$

where the left-hand side is the sum of a finite or countable number of terms.

Problems and Complements

1. Write out all of the partitions of $A = \{a, b, c, d\}$ corresponding to the various congruences of the algebra $\langle A, F \rangle$ where F is a unary operation defined by the array

$$\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}.$$

2. Let $\mathfrak{D}$ be the direct product of models $\mathfrak{M}_1 = \langle \{0, 1\}, \leqq \rangle$ and $\mathfrak{M}_2 = \langle \{2, 3\}, \leqq \rangle$ and let $\varrho_i (i = 1, 2)$ be the kernel congruences in $\mathfrak{D}$ corresponding to the projections $\pi_i : \mathfrak{D} \to \mathfrak{M}_i$. Show that each of the natural mappings $\mathfrak{D}/\varrho_1 \to \mathfrak{M}_1$ and $\mathfrak{D}/\varrho_2 \to \mathfrak{M}_2$ is an isomorphism.

3. If at least one of the cardinals $\mathfrak{a}$ or $\mathfrak{b}$ is infinite, then $\mathfrak{a}\mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b})$ (Hausdorff [18], Section 1.5).

Chapter II

Classical Algebras

3. Groupoids and Groups

3.1 Groupoids and Semigroups. A *groupoid* is an algebra of type $\langle 2 \rangle$. As a rule, the basic operation in a groupoid is called multiplication and the groupoid is said to be *multiplicative*. The product of elements x and y in a groupoid is denoted by $x \cdot y$ or simply xy. Sometimes the basic operation is denoted by $+$ and is called addition. The groupoid is then said to be *additive*.

Let $\mathfrak{G} = \langle G, \cdot \rangle$ be a multiplicative groupoid. An element e of G is a *right identity* for $\mathfrak{G}$ if $xe = x$ for every x in G. A left identity is defined in a similar way. An element e of G is a *two-sided identity* in $\mathfrak{G}$ or simply an *identity* if it is both a left and right-handed identity for $\mathfrak{G}$. No groupoid may have more than one identity. For if $xe = ex = x$ and $xe' = e'x = x$ for all x in G, then $e' = e'e = e$. In an additively written groupoid $\langle G, + \rangle$, the element corresponding to the identity is called the *zero element* and is denoted by 0.

A subalgebra (see Section 2.3) of a groupoid is termed a *subgroupoid*. Observe that a subgroupoid of a groupoid $\mathfrak{G}$ with identity e need not have an identity. For example, $\langle \{1, 2, 3, \ldots\}, \cdot \rangle$ is a groupoid with identity 1. But its subgroupoid $\langle \{2, 4, 6, \ldots\}, \cdot \rangle$ has no identity.

A groupoid is said to be *idempotent* if $xx = x$ for all x in the groupoid.

A groupoid $\mathfrak{G}$ is *commutative* if $xy = yx$ for all x and y in $\mathfrak{G}$.

If the associative law holds for any elements x, y and z in $\mathfrak{G}$: $x(yz) = (xy)z$, then $\mathfrak{G}$ is an *associative groupoid* or *semigroup*. Subalgebras of semigroups are termed *subsemigroups*.

An important example of this is the semigroup of mappings of a set into itself (see Section 1.3). The set of mappings of a fixed set A into itself is known as the *symmetric semigroup* on A. It will be denoted by S_A or $S_{|A|}$.

Theorem 1. *Every semigroup $\mathfrak{G} = \langle G, \cdot \rangle$ is isomorphic to a subsemigroup of the symmetric semigroup on G or on G augmented by an auxiliary element.*

Suppose first that $\mathfrak{G}$ has an identity e. With each element $g \in G$, associate a mapping T_g of G into itself such that $xT_g = xg$ $(x \in G)$.

Since

$$xT_{gh} = x(gh) = (xg)h = (xT_g)\,T_h = xT_gT_h,$$

it follows that $T_{gh} = T_gT_h$. Hence, the correspondence $g \to T_g$ is a homomorphism of $\mathfrak{G}$ onto a subsemigroup of S_G. Since $eT_g = g$ and $eT_h = h$, distinct elements of G map into distinct mappings of S_G. Thus the correspondence $g \to T_g$ is an isomorphism of $\mathfrak{G}$ onto a subsemigroup of S_G.

If $\mathfrak{G}$ has no identity, we adjoin to $\mathfrak{G}$ an auxiliary element $e \notin G$ and we define $ee = e$ and $eg = ge = g$ for $g \in G$. The new set $G' = G \cup \{e\}$ is a semigroup under the multiplication just defined and it contains $\mathfrak{G}$ as a subsemigroup. The first part of the proof now applies to $\mathfrak{G}'$ and hence to $\mathfrak{G}$ augmented by the auxiliary element.

If a semigroup $\mathfrak{G}$ is isomorphic to a subsemigroup of a semigroup $\mathfrak{G}'$, then one says that $\mathfrak{G}$ is *isomorphically embeddable* in $\mathfrak{G}'$. Thus, every semigroup $\mathfrak{G}$ is isomorphically embeddable in a symmetric semigroup.

Let us consider a few other examples of isomorphic embeddings of semigroups. Let A be a nonempty set and let G be the family of all nonempty subsets of A. Define multiplication in G by the formula

$$xy = x \cap y \qquad (x, y, \in G).$$

$\langle G, \cdot \rangle$ is clearly a commutative idempotent semigroup. It is called the *semigroup of subsets* of A (with operation intersection).

Theorem 2. *Every commutative idempotent semigroup* $\mathfrak{A} = \langle A, \cdot \rangle$ *is isomorphically embeddable in the semigroup of subsets of* A.

To each element a in A, let

$$\alpha_a = \{b \in A \mid (\exists x)\, ax = b\}.$$

By virtue of the commutativity, we have

$$\alpha_{ab} = \{c \in A \mid (\exists x)\, abx = c\} \subseteq \alpha_a \cap \alpha_b.$$

Conversely, if d belongs to the intersection $\alpha_a \cap \alpha_b$, then there exist elements x and y in A such that $ax = d$ and $by = d$. By the commutativity and idempotence of the semigroup $\mathfrak{A}$, the equality $ax = by$ yields $abx = by = d$ and so $d \in \alpha_{ab}$. Thus $\alpha_{ab} = \alpha_a \cap \alpha_b$, or in other words, the correspondence $a \to \alpha_a$ is a homomorphism of $\mathfrak{A}$ into the semigroup of subsets of A.

Let $\alpha_a = \alpha_b$. Since $bb = b$, we have $b \in \alpha_b = \alpha_a$. Therefore there exists an element x in A such that $ax = b$. In a similar way, there exists an element y in A for which $by = a$. Hence $a = axy$ and

$b = bxy$. This yields

$$a = axy = by \cdot xy = bxy = b.$$

Therefore the homomorphism $a \to \alpha_a$ is actually an isomorphism.

Consider the semigroup $\mathfrak{J}$ with carrier $\{0, 1\}$ and multiplication defined by the table

	0	1
0	0	0
1	0	1

As a simple consequence of Theorem 2, we have the following statement.

Corollary. *Every commutative idempotent semigroup is isomorphically embeddable in some direct power of the semigroup* $\mathfrak{J}$.

Theorem 2 allows us to assume that our semigroup $\mathfrak{A}$ is the semigroup of subsets of A. The elements of A will be denoted by small Latin letters $a, b, \ldots$ and the elements of the semigroup $\mathfrak{A}$, the subsets of A, by small Greek letters $\alpha, \beta, \ldots$ With each element a in A we associate a copy $\mathfrak{J}_a$ of $\mathfrak{J}$ and we form the direct product

$$\mathfrak{D} = \prod_{a \in A} \mathfrak{J}_a.$$

Its elements are the functions $f(a)$ from A into $\{0, 1\}$ (see Section 2.5). To each set $\alpha \subseteq A$, there corresponds a characteristic function $\alpha(a)$ assuming the value 1 for $a \in \alpha$ and the value 0 for $a \notin \alpha$. The correspondence $\alpha \to \alpha(a)$ defines a mapping of $\mathfrak{A}$ into $\mathfrak{D}$. Since the product $\alpha(a) \cdot \beta(a)$ of functions in $\mathfrak{D}$ clearly corresponds to the intersection $\alpha \cap \beta$ of subsets α and β of A, the mapping $\alpha \to \alpha(a)$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{D}$. But the equality of characteristic functions implies the equality of the subsets corresponding to them and so the homomorphism of $\mathfrak{A}$ into $\mathfrak{D}$ is actually an isomorphism, as required.

A semigroup $\mathfrak{G} = \langle G, \cdot, e \rangle$ with distinguished (or principial) identity e (see Section 2.2) is called a *monoid*. One should not confuse a monoid with a semigroup with identity. A subalgebra of a semigroup with identity need not have an identity but a subalgebra of a monoid is always a semigroup with identity.

Semigroups with a single generating element are said to be *cyclic* or *monogenic* and are easy to classify.

Let a be a generating element of cyclic semigroup $\mathfrak{A}$. Every element of $\mathfrak{A}$ is contained in the sequence $a, a^2, a^3, \ldots$ Suppose first that the sequence has two identical terms a^i and a^j $(i < j)$. Let n be the least

of those numbers j for which for some $i < j$ we have $a^i = a^j$. Now let m be the least of those numbers i for which $a^i = a^n$. The numbers m and n are called *indices* of the cyclic semigroup $\mathfrak{A}$ corresponding to the generating element a. If all terms in the sequence $a, a^2, a^3, \ldots$ are distinct, we define $m = n = 1$.

A pair of positive integers $\langle m, n\rangle$ is an *index pair* for a cyclic semigroup $\mathfrak{A}$ if m and n are indices of $\mathfrak{A}$ for some generating element.

Theorem 3. *Each pair of positive integers* $\langle 1,1\rangle$ *and* $\langle m, n\rangle$ $(m < n;$ $m, n = 1, 2, ..)$ *is the index pair for one and, up to isomorphism, only one cyclic semigroup.*

Proof. $\langle 1, 1\rangle$ is the index pair for the multiplicative semigroup of numbers of the form 2^k $(k = 1, 2, \ldots)$ with generating element 2.

Consider a pair $\langle m, n\rangle$, with $1 \leqq m < n$. For k an arbitrary integer at least 1, define

$$f(k) = \min\big(k, m + \text{rest}\,(k - m, n - m)\big),$$

where rest (x, y) is the remainder on dividing x by y. It is easy to verify that $f(k)$ equals k for $k < n$ and equals $m + \text{rest}\,(k - m, n - m)$ for $k \geqq n$. Write rest $(x, n - m)$ in the abridged form $r(x)$. We wish to show that

$$f\big(k + f(j)\big) = f(k + j) \tag{1}$$

for any positive integers k and j.

Observe first that the congruence

$$f(k) \equiv k \,\big(\text{mod}\,(n - m)\big)$$

holds for $k \geqq n$.

If $j < n$, then $f(j) = j$ and equality (1) is trivial. Therefore suppose $j \geqq n$.

The congruence $f(j) \equiv j \,\big(\text{mod}\,(n - m)\big)$ implies the equality of remainders

$$r\big(k + f(j) - m\big) = r(k + j - m).$$

Therefore, (1) holds for $k + f(j) \geqq n$.

Let $k + f(j) < n$. Since $j \geqq n$, we have

$$f(j) \equiv j \,\big(\text{mod}\,(n - m)\big)$$

and

$$m \leqq f(j) \leqq f(j) + k < n.$$

Hence it follows that

$$r(k + j - m) = r\big(k + f(j) - m\big) = k + f(j) - m,$$

or $k + f(j) = m + r(k + j - m)$ and so (1) is proved.

Now define a binary operation on the set $A = \{1, 2, \ldots, n - 1\}$ as follows:

$$i \circ j = f(j + j).$$

By virtue of (1) we have

$$(i \circ j) \circ k = i \circ (j \circ k) = f(i + j + k)$$

for all i, j and k in A. Therefore, the groupoid $\mathfrak{A} = \langle A, \circ \rangle$ is a semigroup. It is easy to show that the semigroup $\mathfrak{A}$ is generated by the element 1 and that m and n are indices for it with respect to 1. Indeed, setting $1^1 = 1$ and $1^{k+1} = 1^k \circ 1$, we have

$$1^k = f(k) = \begin{cases} k & \text{for } k < n, \\ m + \text{rest } (k - m, n - m) & \text{for } k \geqq n. \end{cases}$$

Therefore, $1^1 < 1^2 < \ldots < 1^{n-1}$, but $1^n = 1^m$.

It merely remains for us to show that cyclic semigroups having the same index pairs are isomorphic.

For the pair $\langle 1, 1 \rangle$, the assertion is trivial. Consider pair $\langle m, n \rangle$ with $1 \leqq m < n$. Let m and n be the indices of a cylic semigroup $\mathfrak{B} = \langle B, \cdot \rangle$ corresponding to generating element b and let $\mathfrak{A}$ be the semigroup constructed above with carrier $A = \{1, 2, \ldots, n - 1\}$. By definition of indices, the elements $b, b^2, \ldots, b^{n-1}$ of $\mathfrak{B}$ are all distinct but $b^n = b^m$. Let k be an arbitrary integer greater than n. Dividing $k - m$ by $n - m$, we have $k - m = (n - m)s + r$, where $s \geqq 1$ and $0 \leqq r < n - m$. Hence, $f(k) = m + r$. Since $b^n = b^m$, we have

$$b^m\big(b^{n-m}\big)^s = b^m b^{n-m}\big(b^{n-m}\big)^{s-1} = b^m\big(b^{n-m}\big)^{s-1} = b^m.$$

Thus,

$$b^k = b^m b^{k-m} = b^m\big(b^{n-m}\big)^s b^r = b^{m+r} = b^{f(k)}.$$

Therefore, $B = \{b, b^2, \ldots, b^{n-1}\}$. It remains to show that the mapping $b^i \to 1^i$ $(i = 1, \ldots, n - 1)$ is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$.

But clearly,

$$b^i b^j = b^{i+j} = b^{f(i+j)}, \quad 1^{f(i+j)} = 1^i \circ 1^j.$$

The proof of Theorem 3 is complete.

3.2 Quasigroups and Loops. A groupoid $\mathfrak{G} = \langle G, \cdot \rangle$ is said to be a *quasigroup* if the equations $ax = b$ and $ya = b$ are each solvable in $\mathfrak{G}$ and the solutions are unique. A subgroupoid $\mathfrak{H}$ of a quasigroup $\mathfrak{G}$ is called a *subquasigroup* if for any a and b in $\mathfrak{H}$ each of the equations $ax = b$ and $ya = b$ is solvable in $\mathfrak{H}$.

A *primitive quasigroup* is an algebra $\mathfrak{G} = \langle G, \cdot, /, \setminus \rangle$ of type $\langle 2, 2, 2 \rangle$ in which the following laws hold for all elements x and y:

(I) $(x/y)y = x, \quad y(y \setminus x) = x,$

(II) $(xy)/y = x, \quad y \setminus (yx) = x.$

The operations $/$ and $\setminus$ are called respectively *right* and *left division* in $\mathfrak{G}$.

It is easy to show that every primitive quasigroup is a quasigroup with respect to the multiplication. Indeed, a solution of $ax = b$ is automatically $a \setminus b$. Since the equation $ax = b$ implies that $a \setminus (ax) = a \setminus b$, the element $x = a \setminus b$ will be the only solution of $ax = b$ on account of (II). In similar fashion, the unique solution of $ya = b$ is the element $y = b/a$.

Conversely, let $\mathfrak{G} = \langle G, \cdot \rangle$ be an ordinary quasigroup. Let $a \setminus b$ denote the unique solution of $ax = b$ and let b/a be the unique solution of $ya = b$. The symbols $\setminus$ and $/$ may be regarded as new operations in G. If we set $x = a \setminus b$ in the equation $ax = b$ and $y = b/a$ in $ya = b$, we obtain the identities

$$(b / a)a = b, \quad a(a \setminus b) = b.$$

Since b is the solution of $ax = ab$ and a the solution of $yb = ab$, we also have

$$a \setminus (ab) = b, \quad (ab) / b = a.$$

Thus, the algebra $\mathfrak{G} = \langle G, \cdot, /, \setminus \rangle$ is a primitive quasigroup.

In other words, there is a one-to-one correspondence between quasigroups and primitive quasigroups. But the algebraic properties of quasigroups and primitive quasigroups are different. For example, subalgebras of primitive quasigroups are subquasigroups whereas subalgebras of ordinary quasigroups are merely subgroupoids.

Theorem 1. *A mapping $\varphi: \mathfrak{G} \to \mathfrak{G}'$ of a primitive quasigroup $\mathfrak{G}$ into a primitive quasigroup $\mathfrak{G}'$ is a homomorphism if for any elements x and y in $\mathfrak{G}$,*

$$\varphi(xy) = \varphi(x)\varphi(y). \tag{1}$$

To show this, take elements x and y in $\mathfrak{G}$ and let $z = x/y$. Then $x = zy$ by virtue of Axiom (I). Hence, $\varphi(x) = \varphi(z)\varphi(y)$ and so

$$\varphi(x/y) = \varphi(z) = \varphi(x)/\varphi(y).$$

Similar reasoning shows that φ preserves left division, namely,

$$\varphi(x \setminus y) = \varphi(x) \setminus \varphi(y)$$

for all x and y in G. Therefore φ is a homomorphism.

A quasigroup is said to be *totally symmetric* (or a TS-quasigroup) if

$$(xy)y = x = y(yx)$$

for all x and y in $\mathfrak{G}$.

It is easy to see that *every totally symmetric quasigroup $\mathfrak{G}$ is commutative.* For, suppose x and y are arbitrary elements of $\mathfrak{G}$ and $yx = z$. Then $(yx)x = zx$ and hence $y = zx$. Multiplying both sides of the last equation on the left by z, we obtain $zy = z(zx) = x$. This implies $z = xy$ and so $xy = yx$.

Theorem 2. *A quasigroup $\mathfrak{G}$ is totally symmetric if and only if its three operations of multiplication, left division and right division are coincident.*

Indeed, if

$$xy = x/y = y \setminus x$$

for any x and y in $\mathfrak{G}$, then $\mathfrak{G}$ is commutative*) and hence

$$(xy)y = (xy)/y = x, \quad y(yx) = y/(yx) = x.$$

Conversely, if quasigroup $\mathfrak{G}$ is totally symmetric, then setting $x/y = u$ and $y \setminus x = v$, we have $x = uy$ and $x = yv$. Hence, $u = xy$ and $v = yx$ and because of the commutativity, $u = v$. In other words, $xy = x/y = y \setminus x$.

A quasigroup with identity is called a *loop*. A *primitive loop* is a primitive quasigroup that satisfies besides (I) and (II) the further law

(III) $x/x = y \setminus y$.

A primitive loop is a loop with respect to multiplication since (I) and (III) imply that the quotient

$$e = x/x = y \setminus y$$

is a two-sided identity. Conversely, if $\mathfrak{L}$ is an ordinary loop with identity e, then for any two elements x and y in $\mathfrak{L}$ we have

$$ex = x, \quad ye = y.$$

*) Using (I) and (II), $xy = y\,(y \setminus (xy)) = y((xy)/y) = yx$.

Thus, $x / x = e = y \setminus y$, i.e., $\mathfrak{L}$ is a primitive loop with respect to the operations of multiplication, left division and right division.

Subquasigroups of loops are called *subloops*. For primitive loops, the notion of subloop coincides with the notion of subalgebra. In other words, a nonempty subset of a primitive loop is a subloop if and only if it is closed under all three fundamental operations of a primitive loop: multiplication, left division, and right division.

3.3 Groups. A *group* is an algebra $\mathfrak{G} = \langle G, \cdot, {}^{-1} \rangle$ of type $\langle 2, 1 \rangle$ whose basic operations are related on G by the axioms

(I) $x(yz) = (xy)z$,

(II) $y^{-1}(yx) = x = (xy)y^{-1}$.

Here, y^{-1} is the result of applying the unary operation $^{-1}$ to element y Subalgebras of a group are called *subgroups*.

If we set $e = y^{-1}y$, we have $ex = xe = x$ for all $x \in G$. Therefore the element e is an identity for the groupoid $\langle G, \cdot \rangle$. Since every groupoid has at most one identity (see Sec. 3.1), e is a fixed element of the group not depending on y. If we set $x = e$ in Axiom (II), we have $y^{-1}y = e = yy^{-1}$. Therefore y^{-1} is the *inverse of* y and Axiom (II) is called the *inverse law*.

Each of the equations $ax = b$ and $ya = b$ has a unique solution in a group given by $x = a^{-1}b$ and $y = ba^{-1}$, respectively. To prove this, we merely multiply each side of the first equation on the left by a^{-1} and each side of the second equation on the right by a^{-1} and then we apply the inverse law. Thus a group is a loop under multiplication. A group may obviously be regarded as a primitive loop with operations xy, $x/y = xy^{-1}$ and $y \setminus x = y^{-1}x$.

Conversely, given any associative quasigroup $\langle G, \cdot \rangle$, a unary operation x^{-1} may be defined on it in such a way that the algebra $\langle G, \cdot, {}^{-1} \rangle$ is a group. For, suppose $a \in G$ and e is the solution of $ax = a$. Write an arbtrary element $b \in G$ as $b = ca$ (such a c exists). By the associativity of multiplication, $be = (ca)e = c(ae) = ca = b$. Thus, e is a right identity for the quasigroup. In similar fashion, the solution e' of $ya = a$ is a left identity. Finally, $e' = e'e = e$. Thus, e is the identity. If we now let a^{-1} stand for the solution of $ax = e$, we have $aa^{-1}a = a$. This implies $a^{-1}a = e$ since $ay = a$ has a unique solution. Hence, $a^{-1}a = aa^{-1} = e$ and so $\langle G, \cdot, {}^{-1} \rangle$ is a group.

The above discussion suggests that a group could be viewed as an algebra with one basic multiplication operation since the inverse operation is definable (though not expressible) in terms of multiplication. But then a subalgebra of a group would be a subsemigroup whereas in

group theory, an important part is played by the subalgebras closed under multiplication and the inverse operation.

Nevertheless a group can viewed as an algebra of type $\langle 2\rangle$ with one basic binary operation except not multiplication but division $\div$ expressed in terms of multiplication and the inverse operation by means of the formula

$$x \div y = xy^{-1}. \tag{1}$$

Group multiplication and its inverse are then expressed in terms of division through

$$x^{-1} = (x \div x) \div x, \tag{2}$$

and

$$xy = x \div y^{-1} = x \div \big((y \div y) \div y\big). \tag{3}$$

From the group axioms it follows that division satisfies the following laws:

$$(x \div z) \div (y \div z) = x \div y, \tag{4}$$

$$x \div (y \div y) = x, \tag{5}$$

$$(x \div x) \div (y \div z) = z \div y. \tag{6}$$

Conversely, if the operation $\div$ satisfies (4)—(6) in a groupoid $\langle G, \div\rangle$, then by defining multiplication and the inverse operation in G by means of (2) and (3), we obtain a group $\langle G, \cdot, {}^{-1}\rangle$ in which the operation $\div$ can be expressed in terms of multiplication and the inverse operation through (1).

Indeed, (4) and (5) yield

$$(x \div y) \div y^{-1} = (x \div y) \div \big((y \div y) \div y\big) = x \div (y \div y) = x,$$

and

$$(x \div y^{-1}) \div y = (x \div y^{-1}) \div \big((y \div y) \div y^{-1}\big) = x \div (y \div y) = x.$$

From (6) results

$$(x \div y)^{-1} = \big((x \div y) \div (x \div y)\big) \div (x \div y) = y \div x.$$

Hence, multiplication is associative:

$$\begin{aligned} x(yz) = x \div (y \div z^{-1})^{-1} = x \div (z^{-1} \div y) = \big((x \div y^{-1}) \div y\big) \div (z^{-1} \div y) \\ = (x \div y^{-1}) \div z^{-1} = (xy)z. \end{aligned}$$

From (2), (6) and (5) we have

$$\begin{aligned}(x^{-1})^{-1} &= \big[\big((x \div x) \div x\big) \div \big((x \div x) \div x\big)\big] \div \big((x \div x) \div x\big) \\ &= x \div (x \div x) = x,\end{aligned}$$

This yields

$$\begin{aligned}yy^{-1} &= y \div (y^{-1})^{-1} = y \div y, \\ (xy)y^{-1} &= x(yy^{-1}) = x \div (y \div y) = x, \\ y^{-1}(yx) &= y^{-1} \div (y \div x^{-1})^{-1} = y^{-1} \div (x^{-1} \div y) \\ &= \big((y \div y) \div y\big) \div (x^{-1} \div y) = (y \div y) \div x^{-1} \\ &= \big(x^{-1} \div (y \div y)\big)^{-1} = (x^{-1})^{-1} = x.\end{aligned}$$

Thus the algebra $\langle G, \cdot, {}^{-1}\rangle$ is a group and the original operation $\div$ on G is expressed in terms of multiplication and the inverse operation through (1).

Theorem 1. *If a mapping φ of a group $\mathfrak{G} = \langle G, \cdot, {}^{-1}\rangle$ into a group $\overline{\mathfrak{G}} = \langle \overline{G}, \cdot, {}^{-1}\rangle$ preserves multiplication, i.e., $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y in G, then φ carries the identity in $\mathfrak{G}$ into the identity in $\overline{\mathfrak{G}}$ and so is a homomorphism.*

To this end, let e be the identity in $\mathfrak{G}$ and let $\bar{e} = \varphi(e)$. Since $ee = e$, $\varphi(e)\varphi(e) = \varphi(e)$, i.e., $\bar{e}\bar{e} = \bar{e}$. This implies that $\bar{e}$ is the identity in $\overline{\mathfrak{G}}$. Let $x \in G$ and let $y = x^{-1}$. Since $xy = e$, $\varphi(x)\varphi(y) = \varphi(e) = \bar{e}$. Hence $\varphi(x^{-1}) = \varphi(y) = \big(\varphi(x)\big)^{-1}$ and so φ is a homomorphism of $\mathfrak{G}$ into $\overline{\mathfrak{G}}$.

For each fixed element a and any element x of group $\mathfrak{G}$, $\varphi_a\colon\ x \to a^{-1}xa$ is a one-to-one mapping of $\mathfrak{G}$ onto itself since each element $y \in G$ has a unique pre-image $ayа^{-1}$ in $\mathfrak{G}$. Furthermore, we have

$$\varphi_a(xy) = a^{-1}(xy)a = a^{-1}xa \cdot a^{-1}ya = \varphi_a(x)\varphi_a(y).$$

Hence the mapping φ_a is an automorphism of $\mathfrak{G}$.

φ_a is said to be the *inner* automorphism of group $\mathfrak{G}$ generated by element a.

A subgroup N of $\mathfrak{G}$ is said to be *invariant* or *normal* in $\mathfrak{G}$ if it is mapped into itself under any inner automorphism of $\mathfrak{G}$.

If H is a subgroup of $\mathfrak{G}$, the set of elements of the form ha, where h is an arbitrary element of H and a is a fixed element of G, is a *left coset* of H in $\mathfrak{G}$ and is denoted by Ha. A right coset aH is defined in a similar way. A subgroup H is clearly normal in $\mathfrak{G}$ if and only if

$$Ha = aH$$

for any $a \in \mathfrak{G}$.

For any subgroup H of $\mathfrak{G}$, every pair of left cosets Ha and Hb either have no elements in common or coincide. For, suppose $Ha \cap Hb \neq \varnothing$. Then letting c be an element in the intersection, we have

$$c = h_1 a = h_2 b$$

for some h_1 and h_2 in H. Hence $a = h_1^{-1} h_2 b$ and so $Ha \subseteqq Hb$. By the same token, $b = h_2^{-1} h_1 a$ and so $Hb \subseteqq Ha$. Thus, $Ha = Hb$, as asserted.

Therefore, every group $\mathfrak{G}$ is representable in terms of the disjoint left cosets of a subgroup H by

$$\mathfrak{G} = H + Ha_2 + Ha_3 + \ldots,$$

where the plus sign stands for set-theoretic union. A decomposition of a group $\mathfrak{G}$ into a set-theoretic sum of right cosets of subgroup H is obtained in a similar way.

A subgroup H of $\mathfrak{G}$ is said to be of *finite index in* $\mathfrak{G}$ if the number of cosets of H in $\mathfrak{G}$ is finite. The number of cosets is called the *index of* H *in* $\mathfrak{G}$.

Theorem 2. *The equivalence classes with respect to any congruence* θ *on group* $\mathfrak{G}$ *are the cosets of some normal subgroup* N *of* $\mathfrak{G}$. *Conversely, to each normal subgroup* N *in* $\mathfrak{G}$ *there corresponds a congruence on* $\mathfrak{G}$ *whose equivalence classes are the cosets of* N *in* $\mathfrak{G}$.

To see this, let

$$N = \{x \mid x \in G, x \theta e\},$$

where e is the identity of $\mathfrak{G}$. Then N is a subgroup of $\mathfrak{G}$ since if $x \theta e$ and $y \theta e$, we have $x^{-1} \theta e$ and $xy \theta e$ by definition of congruence. The subgroup N is normal in $\mathfrak{G}$ since if $x \theta e$, then $g^{-1} x g \theta e$. It is easy to show that $[a]_\theta = Na$ for arbitrary a in $\mathfrak{G}$.

Conversely, let N be a normal subgroup of $\mathfrak{G}$. The relation

$$a \theta b \Leftrightarrow Na = Nb \quad (a, b \in G)$$

is clearly a congruence on $\mathfrak{G}$ and $[a]_\theta = Na$ for any a in G.

Thus, there is a one-to-one correspondence between congruences and normal subgroups of a group. Therefore the quotient group of $\mathfrak{G}$ by congruence θ is also referred to as the quotient group of $\mathfrak{G}$ by the (corresponding) normal subgroup N and is denoted by $\mathfrak{G}/N$. By definition of quotient system (see Section 2.4), we have

$$Na \cdot Nb = Nab, \quad (Na)^{-1} = Na^{-1}. \tag{7}$$

Let $\mathfrak{S} = \langle S, \cdot \rangle$ be a semigroup with identity e. An element $a \in S$ is *invertible* if there is an element $a^{-1} \in S$ such that $aa^{-1} = a^{-1}a = e$. Since each invertible a in S has a unique inverse a^{-1}, the operation $^{-1}$ of taking inverses is well-defined on the whole set G of invertible elements of $\mathfrak{S}$. G is also closed under multiplication and hence $\langle G, \cdot, ^{-1} \rangle$ is a group. It is called the *group of invertible elements* of semigroup $\mathfrak{S}$ with identity.

The group of invertible elements of a symmetric semigroup S_A (see Section 3.1) is termed the *symmetric group* and is also denoted by S_A or $S_{|A|}$.

If $\langle G, \cdot, ^{-1} \rangle$ is a group, then for each $g \in G$, the mapping $T_g: x \to xg$ $(x \in G)$ of G into itself is an invertible element of the symmetric semigroup S_G. By Theorem 1 of Section 3.1 and Theorem 1 of the current subsection, the following holds.

Theorem 3. *Every group $\langle G, \cdot, ^{-1} \rangle$ is isomorphically embeddable in its symmetric group S_G.*

If $\mathfrak{A} = \langle A, \Omega \rangle$ is an arbitrary algebraic system, all of its automorphisms (see Section 2.2) comprise a group under the operations of composition and taking of inverses considered in Section 1.3. It is called the *automorphism group* of $\mathfrak{A}$ and is denoted by Aut $\mathfrak{A}$. Subgroups of Aut $\mathfrak{A}$ are called simply *automorphism groups* of the algebraic system $\mathfrak{A}$. In particular, the inner automorphisms of a group $\mathfrak{G}$ comprise a subgroup of Aut $\mathfrak{G}$ called the *group of inner automorphisms of* $\mathfrak{G}$.

The structurally simplest subgroups of a group $\mathfrak{G}$ are its cyclic subgroups, i.e., subgroups generated by a single element. They are formed as follows. Take an arbitrary element g in G not equal to the identity e. Define

$$g^0 = e, \quad g^{n+1} = g^n g, \quad g^{-n} = (g^{-1})^n \qquad (n = 0, 1, 2, \ldots).$$

An immediate consequence of this is that

$$g^m g^n = g^{m+n} \quad \text{and} \quad (g^m)^n = g^{mn}$$

for any integers m and n. The collection of elements of the form g^n $(n = 0, \pm 1, \pm 2, \ldots)$ is the subgroup of $\mathfrak{G}$ generated by element g. The least positive integer n for which $g^n = e$ is called the *order* of g. If $g^n \neq e$ for any $n = 1, 2, \ldots$, then g is said to be of *infinite order*. The order of the identity element e is taken to be zero.

A cyclic subgroup of a group $\mathfrak{G}$ generated by an element g of finite order $n \geqq 1$ comprises the elements

$$e, g, g^2, \ldots, g^{n-1}.$$

For, if we take an arbitrary integer m and divide it by n, we obtain $m = nq + r$, $0 \leqq r < n$, and so $g^m = g^r$. All of the powers $g^0 = e, g,$

$g^{-1}, g^2, g^{-2}, \ldots$ in a cyclic group generated by an element g of infinte order are obviously distinct.

An example of an infinite cyclic group is the group $\langle\{0, 1, -1, 2, -2, \ldots\}, +, -\rangle$ generated by the element 1. For fixed $n \geqq 1$, all n-th roots of 1 form a cyclic group of finite order generated by any primitive n-th root of unity (under multiplication and inversion of root extraction).

Let p be a fixed prime number. Each cyclic group $\mathfrak{A}_n$ of order p^n generated by an element a has a cyclic subgroup $\{a^p\}$ of order p^{n-1} generated by a^p. This defines a direct spectrum $\mathfrak{A}_1 \to \mathfrak{A}_2 \to \ldots$ whose limit (see Section 2.3) is called a *group of type* p^∞. This group is isomorphic to the multiplicative group of all n-th roots of 1 with $n = 1, p, p^2, \ldots$ and it will be denoted by Z_{p^∞}.

A group $\mathfrak{G}$ is *commutative* or *abelian* if $xy = yx$ for x, y in $\mathfrak{G}$. Cyclic groups and the groups of type p^∞ are the simplest examples of abelian groups.

Let $\mathfrak{G} = \langle G, \cdot, ^{-1}\rangle$ be an arbitrary group and let a and b be elements of G. The element

$$[a, b] = a^{-1}b^{-1}ab$$

is called the *commutator* of a and b. Since

$$ab = ba[a, b],$$

group multiplication is abelian if and only if $[a, b] = e$.

The set K of all commutators in a group $\mathfrak{G}$ is closed under the taking of inverses since

$$[a, b]^{-1} = b^{-1}a^{-1}ba = [b, a].$$

The subgroup generated by the set K*) obviously consists of all possible products of elements of K. It is called the *commutator* or *derived subgroup* of $\mathfrak{G}$ and will be denoted by G'.

Since an inner automorphism φ_g of a group $\mathfrak{G}$ satisfies

$$[a, b]\varphi_g = g^{-1}[a, b]g =$$
$$= (g^{-1}ag)^{-1}(g^{-1}bg)^{-1}(g^{-1}ag)\,(g^{-1}bg) = [a\varphi_g, b\varphi_g],$$

G' is a normal subgroup of G. By (7), $[G'a, G'b] = G'$ and therefore $\mathfrak{G}/G'$, the quotient group of $\mathfrak{G}$ by its commutator subgroup, is abelian.

It is easy to show that *the commutator subgroup of* $\mathfrak{G}$ *is contained in every normal subgroup* N *of* $\mathfrak{G}$ *whose corresponding quotient group* $\mathfrak{G}/N$ *is abelian.*

*) For the definition of generating set, see Section 2.3.

An element z of a group $\mathfrak{G}$ is said to be central in $\mathfrak{G}$ if it commutes with each element g of the group, i.e., if $[z, g] = e$ for all g in $\mathfrak{G}$. The set Z of all central elements of $\mathfrak{G}$ is clearly closed under the basic operations of multiplication and the taking of inverses in $\mathfrak{G}$ and so is a subgroup of $\mathfrak{G}$. It is called the *center* of $\mathfrak{G}$. It is easy to see that Z is an abelian group.

Since an inner automorphism φ_g of $\mathfrak{G}$ leaves each central element z of $\mathfrak{G}$ fixed, namely,

$$z\varphi_g = g^{-1}zg = z,$$

the subgroup Z is normal in G.

A group $\mathfrak{G}$ is thus abelian if and only if at least one of the following hold:

$$G' = \{e\}, \quad Z = G.$$

A group $\mathfrak{G}$ is called. *metabelian* if its commutator subgroup G' is contained in its center Z

Consider an arbitrary group $\mathfrak{G}$. Put $\mathfrak{G}^{(0)} = \mathfrak{G}$ and let $\mathfrak{G}^{(n+1)}$ be the commutator subgroup of $\mathfrak{G}^{(n)}$. This yields a chain of successive commutator subgroups $\mathfrak{G}^{(0)} \supseteq \mathfrak{G}^{(1)} \supseteq \mathfrak{G}^{(2)} \supseteq \ldots$ $\mathfrak{G}$ is said to be *solvable* if $\mathfrak{G}^{(n)} = \{e\}$ for some integer $n \geqq 0$*) and *locally solvable* if all finitely generated subgroups of $\mathfrak{G}$ are solvable.

For any elements $x_1, \ldots, x_{n+1}$ of $\mathfrak{G}$ ($n \geqq 2$), put

$$[x_1, \ldots, x_{n+1}] = \big[[x_1, \ldots, x_n], x_{n+1}\big].$$

A *lower central chain of* $\mathfrak{G}$ is the descending chain $\mathfrak{G}^{(1)} \supseteq \mathfrak{G}^{(2)} \supseteq \mathfrak{G}^{(3)} \supseteq \ldots$ of subgroups $\mathfrak{G}^{(n)}$ each generated by all the commutators of the form $[g_1, \ldots, g_n]$ $(g_1, \ldots, g_n \in \mathfrak{G})$. $\mathfrak{G}$ is said to be *nilpotent* if $\mathfrak{G}^{(n)} = \{e\}$ for some integer $n \geqq 1$. Metabelian groups are the only nilpotent groups for which $\mathfrak{G}^{(3)} = \{e\}$.

Problems and Complements

1. Every abelian group having a finite number of n generators is decomposable into a direct product of at most n cyclic subgroups.

2. To each nonempty subset I of $N = \{2, 3, \ldots\}$, define a semigroup $\mathfrak{A}_I$ by means of generators a and b satisyfing

$$a^i b a^i \cdot b^i a b^i = b^i a b^i \cdot a^i b a^i \quad (i \in I).$$

Prove that for $I \neq I'$, the semigroups $\mathfrak{A}_I$ and $\mathfrak{A}_{I'}$ are not isomorphic (L. A. Bokut').

3. For any cardinal $\mathfrak{n} \geqq 2$, the set of all nonisomorphic semigroups with $\mathfrak{n}$ generators has cardinality $2^{\mathfrak{n}+\aleph_0}$. A similiar result also holds for groups (see Neumann [53]).

---

*) The descending chain of commutator groups is a particular case of a normal chain with solvability corresponding to its being a normal series [*Trans.*].

4. Rings and Fields

4.1 Rings. A *ring* is an algebra of type $\langle 2, 1, 2\rangle$ in which the basic operations: $x + y$, $-x$ and xy are connected by the following axioms:

(I) $x + (y + z) = (x + y) + z$,

(II) $x + y = y + x$,

(III) $(-x) + (x + y) = y$,

(IV) $(x + y)z = xz + yz$, $x(y + z) = xy + xz$.

Subalgebras of rings are called *subrings*.

Axioms (I), (II), and (III) show that the algebra $\langle K, +, -\rangle$ corresponding to each ring $\mathfrak{R} = \langle K, +, -, \cdot\rangle$ is an *additive abelian group.**) The zero element of the group is called the zero element of the ring $\mathfrak{R}$. It is easy to see that every abelian group $\langle A, +, -\rangle$ plays the role of the additive group of some ring. One example is the ring $\langle A, +, -, \cdot\rangle$ with the trivial multiplication $xy = 0$ for all x and y in A.

If 0 is the zero element and x and y arbitrary elements of a ring, we obtain

$$xy = x(y + 0) = xy + x0, \quad yx = (y + 0)x = yx + 0x,$$

and so

$$x0 = 0x = 0.$$

We further have

$$0 = x\big(y + (-y)\big) = xy + x(-y),$$
$$0 = \big(x + (-x)\big)y = xy + (-x)y,$$

and therefore

$$x(-y) = (-x)y = -xy, \qquad (-x)\,(-y) = xy.$$

Using the distributive laws (IV), we can apply induction to derive the general distributive law

$$(x_1 + x_2 + \ldots + x_m)\,(y_1 + y_2 + \ldots + y_n) = \sum_{i,j} x_i y_j,$$

which in turn implies the identity

$$(mx)\,(ny) = (mn)\,(xy)$$

for any integers m and n.

---

*) Notice that $\langle K, \cdot\rangle$ is merely a groupoid. This is a little different from the standard definition [*Trans.*].

A subring I of a ring $\mathfrak{R}$ is called an *ideal in* $\mathfrak{R}$ if ax and xa belong to I whenever a is in I and x is in $\mathfrak{R}$.

Since every ideal of a ring $\mathfrak{R}$ is a subgroup of the additive group of $\mathfrak{R}$, we can speak of the cosets $I + x$ of I in $\mathfrak{R}$ and of the decomposition of K with respect to I.

Theorem 1. *There is a one-to-one correspondence between ideals and congruences on a ring* $\mathfrak{R}$ *under which the cosets of an ideal in* $\mathfrak{R}$ *coincide with the equivalence classes of the congruence corresponding to this ideal.*

Let θ be a congruence on $\mathfrak{R}$. Define

$$I = \{a \in K \mid a \theta 0\}.$$

It is easy to show that I is an ideal in $\mathfrak{R}$ and $[x]_\theta = I + x$ for all $x \in \mathfrak{R}$. The correspondence $\theta \to I$ is one-to-one. For, if I is an arbitrary ideal in $\mathfrak{R}$, then on setting

$$x \theta y \Leftrightarrow I + x = I + y \quad (x, y, \in K),$$

we have a congruence θ for which the ideal I coincides with the set of elements comparable with 0.

In view of the existing correspondence, the quotient ring $\mathfrak{R}/\theta$ is also customarily denoted by $\mathfrak{R}/I$, where I is the ideal corresponding to θ.

If $\mathfrak{R} = \langle K, +, -, \cdot \rangle$ is a ring and the groupoid $\langle K, \cdot \rangle$ possesses an identity, then the identity is called the ring identity and is denoted by 1 or e.

An element a of a ring $\mathfrak{R}$ with identity is said to be invertible in $\mathfrak{R}$ if there is an element a^* in $\mathfrak{R}$ such that $aa^* = a^*a = 1$. The element a^* is the *inverse of* a and is denoted by a^{-1}.

An element a of a ring $\mathfrak{R}$ is called a *left zero divisor* if $a \neq 0$ and there exists a nonzero element x in K such that $ax = 0$. A *right zero divisor* is defined in a similar way. A ring having no left nor right zero divisors is said to be a *ring without zero divisors*.

Holding in every ring $\mathfrak{R}$ without zero divisors are the cancellation laws

$$ax = ay \Rightarrow x = y, \quad xa = ya \Rightarrow x = y$$

$$(a, x, y \in K, \ a \neq 0)$$

$\mathfrak{R}$ is called a *ring with division* if for any a and b in K, $a \neq 0$, each of the equations

$$ax = b, \quad ya = b$$

has a solution in K. If each of the equations has a unique solution when $a \neq 0$ and $\mathfrak{R}$ has an identity, then $\mathfrak{R}$ is called a *division ring*.*) In other words, a division ring is a ring with division, with an identity and no zero divisors. A subring of $\mathfrak{R}$ is also a division ring, called a *subdivision ring* of $\mathfrak{R}$.

From the given definitions, it follows that a subring T of a division ring $\mathfrak{R}$ is a subdivision ring if and only if T has an identity and is a ring with division.

The set of all nonzero elements of a division ring comprises a multiplicative loop (see Section 3.2). Thus it follows in particular that a division ring $\mathfrak{R}$ has no ideals apart from $\{0\}$ and $\mathfrak{R}$ and so is *simple*. Indeed, if a is a nonzero element of an ideal I in $\mathfrak{R}$, then I contains the element $a(a \setminus 1) = 1$. This implies $I = \mathfrak{R}$.

It should be emphasized that while addition $x + y$, multiplication xy and the taking of inverses $-x$ are defined throughout a division ring, the loop operations of left and right division ($\setminus$ and $/$) are only partial in a division ring. Therefore a division ring is merely a partial algebra under all of the operations $+, -, \cdot, \setminus, /$.

A ring $\mathfrak{R}$ is *associative* if the associative law holds for any of its elements x, y and z, namely,

$$(xy)z = x(yz).$$

The *associator* of elements x, y and z in an arbitrary ring $\mathfrak{R}$ is the ring element

$$[x, y, z] = (xy)z - x(yz)$$

A ring is clearly associative if and only if the associator of any of its elements x, y, and z is the zero element.

A ring $\mathfrak{R}$ is called *alternative* if its elements satisfy the alternative law

$$[x, x, y] = 0, \quad [x, y, x] = 0, \quad [y, x, x] = 0.$$

It is easy to show that each of these relations is a consequence of the remaining two. Thus only two of the three need be retained in the definition of alternative ring. For example, if the first two parts of the alternative law hold in ring $\mathfrak{R}$, then $[x + y, x + y, x] = 0$ for any x and y in $\mathfrak{R}$. Since an associator is distributive with respect to each

*) Notice the lack of associativity, counter to the customary definition of division ring. Actually there exists a separate term in the Russian for such a structure [*Trans.*].

argument, we have

$$[x, x, x] + [y, x, x] + [x, y, x] + [y, y, x] = 0,$$

which implies $[y, x, x] = 0$.

A ring $\mathfrak{R}$ is called a *Lie ring* if both the relation $xx = 0$ and Jacobi identity

$$x(yz) + y(zx) + z(xy) = 0$$

hold for any of its element x, y and z.

It thus follows that a Lie ring cannot have an identity element and so cannot be a division ring.

A ring $\mathfrak{R}$ is *commutative* if its elements obey the commutative law

$$xy = yx,$$

and *anticommutative*, if they obey the law

$$xy = -yx.$$

If x and y are arbitrary elements of a Lie ring, we have $(x + y) \times (x + y) = 0$ and hence $xy + yx = 0$. In other words, *every Lie ring is anticommutative.*

An associative ring satisfying the idempotent law

$$xx = x$$

is called a *Boolean ring.*

It is easy to show that *a Boolean ring is commutative and satisfies*

$$x + x = 0.$$

Indeed, the law $xx = x$ implies that $(x + y)(x + y) = x + y$ identically. Removing parentheses on the left-hand side of this equation, we obtain $x + y + xy + yx = x + y$ or $xy + yx = 0$. If we set $y = x$, we have $x + x = 0$. Now using the equation $x = -x$, we then obtain $xy - yx = 0$, i.e., $xy = yx$.

A *field* is an associative commutative division ring. Inversely, a nonassociative noncommutative field is a division ring. Let us agree to call it a *skew field* or *s-field.*

The equation $ax = b$ $(a \neq 0)$ has a unique solution in a field $\mathfrak{F}$. It is denoted by the fraction $\frac{b}{a}$ and is called the quotient of b by a. All of the usual rules governing the arithmetic operations on numerical fractions remain valid for an arbitrary field. In particular,

$$\frac{ac}{bc} = \frac{a}{b} \quad (c \neq 0), \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The set of nonzero elements of a field comprises an abelian group under multiplication xy and the taking of inverses $x^{-1} = \frac{1}{x}$. Since the element 0^{-1} is undefined, taking inverses is a partial operation in a field.

Each subring F of a field $\mathfrak{F}$ which is also a field is a *subfield* of $\mathfrak{F}$. $\mathfrak{F}$ in turn is said to be an *extension field* of F. If M is an arbitrary subset of $\mathfrak{F}$ and F is a subfield of $\mathfrak{F}$, it is customary to let $F(M)$ denote the intersection of all subfields of $\mathfrak{F}$ containing F and M. $F(M)$ is clearly a subfield of $\mathfrak{F}$. The process of obtaining an extension field by adjoining the elements of M to F is known as field adjunction.

Let F be the intersection of all sub-skew fields of a skew field $\mathfrak{K}$. From the definition of division ring, it immediately follows that F is a skew field. For k a nonzero integer, the equation $ke \cdot x = e$ has a unique solution in F which we denote by $(ke)^{-1}$. Clearly, $ke \cdot (ke)^{-1} = (ke)^{-1} \cdot ke = e$. Let a and b be arbitrary elements of F. Then

$$(ke \cdot a)\,(pe \cdot b) = ka \cdot pb = (kp)ab.$$

Therefore the equations $ke \cdot x = e$ and $pe \cdot y = e$ $(k \neq 0, p \neq 0)$ imply that $kpe \cdot xy = e$, i.e.,

$$(ke \cdot pe)^{-1} = (ke)^{-1}(pe)^{-1}.$$

Setting $b = e$, we also find

$$(ke \cdot a) \cdot pe = ke \cdot (a \cdot pe) = kpa.$$

Therefore if we multiply $ke \cdot pe = pe \cdot ke$ $(k \neq 0)$ on the left and on the right by $(ke)^{-1}$, we obtain

$$(ke)^{-1}(pe) = (pe)\,(ke)^{-1}.$$

Now set

$$\frac{me}{ke} = me \cdot (ke)^{-1}.$$

We then have

$$\frac{me}{ke} + \frac{ne}{pe} = \frac{(mp + kn)e}{kpe}, \quad \frac{me}{ke} \cdot \frac{ne}{pe} = \frac{mne}{kpe}.$$

Thus the elements of the form $me \cdot (ke)^{-1}$ comprise a field which has to coincide with F. It is called the *prime* subfield of the skew field $\mathfrak{K}$.

If $me \neq ne$ for $m \neq n$, the correspondence $me \cdot (ke)^{-1} \to mk^{-1}$ $(k \neq 0)$ is an isomorphic mapping of the field F onto the field of rational

numbers. In that case, the given skew field $\mathfrak{K}$ is said to *have characteristic zero.*

If for some $m > n > 0$, we have $me = ne$, then $(m - n)e = 0$. Since $e \neq 0$, $m - n$ must exceed 1. Among the integers $t > 1$ such that $te = 0$, there has to be a least one. It is called the *characteristic* of the skew field $\mathfrak{K}$.

Theorem 2. *The characteristic of a skew field is either zero or a prime.*

To show this, let p be the characteristic of a skewfield such that $p = p_1 p_2$ and $1 < p_1 < p$ and $1 < p_2 < p$. Then $p_1 e \cdot p_2 e = pe = 0$ and so either $p_1 e = 0$ or $p_2 e = 0$. But this is impossible because p is the least number with that property.

If the characteristic of a skew field $\mathfrak{K}$ is prime p, then the prime subfield F consists of the elements

$$0, e, 2e, \ldots, (p - 1)e.$$

For, from $pe = 0$ it follows that $me + ne = r_1 e$ and $me \cdot ne = r_2 e$, where r_1 and r_2 are the remainders when $m + n$ and mn are respectively divided by p. Further, if $me \neq 0$, then m and p are relatively prime and therefore there exist integers x and y such that $mx + py = 1$. But then $e = xme + ype$, i.e., $xe \cdot me = e$ or $(me)^{-1} = xe$. Thus, each element of F is representable in the form re with $0 \leqq r < p$.

Observe that in a skew field of zero characteristic, the solution of $mx = 0$ $(m \neq 0)$ is $x = 0$. But in a skew field of prime characteristic p, each element x satisfies $px = pe \cdot x = 0$ and so $mx = 0$ has the solution $x = 0$ only when m is not divisible by p.

From our university courses in modern algebra, we know how to construct a ring of polynominals $\mathfrak{F}[x]$ over a given field $\mathfrak{F}$ in one indeterminate x as well as a ring of polynominals $\mathfrak{F}[x_1, \ldots, x_\xi, \ldots]$ in arbitrarily many indeterminates $x_1, \ldots, x_\xi, \ldots$ We also know that $\mathfrak{F}[x_1, \ldots, x_\xi, \ldots]$ is an associative-commutative ring with no zero divisors and that it is embeddable in the field $\mathfrak{F}(x_1, \ldots, x_\xi, \ldots)$ of quotients of polynomials. It will be assumed that the notions of root of a polynomial and irreducible polynomial are known as well as the theorem on the existence of a splitting field for every (nonconstant) polynomial with coefficients in an arbitrary field. It will also be assumed that the reader is acquainted with the technique of adjunction of indeterminates $x_1, \ldots, x_\xi, \ldots$ to a given field or that of the roots $a, b, \ldots$ of polynomials in $\mathfrak{F}[x]$. The corresponding extension fields will be denoted by $\mathfrak{F}(x_1, \ldots, x_\xi, \ldots)$ and $\mathfrak{F}(a, b, \ldots)$, respectively.

An ideal I is said to be *maximal* in a ring $\mathfrak{K}$ if $I \neq \mathfrak{K}$ and if for every ideal I_1 in $\mathfrak{K}$ containing I, either $I_1 = \mathfrak{K}$ or $I_1 = I$.

Theorem 3. *If $\mathfrak{R}$ is a commutative-associative ring with identity e and I is a maximal ideal in $\mathfrak{R}$, then the quotient ring $\mathfrak{R}/I$ is a field.*

First of all, $\mathfrak{R}/I$ is obviously associative and commutative and has $I + e$ for identity. We therefore merely have to show that $\mathfrak{R}/I$ is a ring with division.

Consider the equation $\bar{x}\bar{a} = \bar{a}\bar{x} = \bar{b}$ in $\mathfrak{R}/I$ with $\bar{a} \neq 0$. The set $I_1 = \{i + ka \mid i \in I, k \in K\}$ is clearly an ideal in $\mathfrak{R}$ containing I and the element $a = 0 + ea$. Since $a \notin I$, $I_1 = \mathfrak{R}$ and so each element of $\mathfrak{R}$ may be written in the form $i + ka$ with $i \in I$ and $k \in \mathfrak{R}$. There exists a $k \in \mathfrak{R}$ for which $b = i + ka$ and so $I + b = I + ka$. In other words, $\bar{b} = \bar{k}\bar{a}$ and this implies $\mathfrak{R}/I$ is a field.

4.2 Algebraically Closed Fields. In the sequel, we shall have occasion to make use of algebraically closed fields for illustrative purposes. For the convenience of the reader, we state the definitions and results needed and we refer him for details to van der Waerden's book [75] (and also Zariski and Samuel [76]).

An extension $\mathfrak{R}$ of a field F is said to be *algebraic over F* if each element of $\mathfrak{R}$ is *algebraic over F*, i.e., it is the root of some polynomial in the ring $F[x]$.

An extension $\mathfrak{R}$ of a field F may be regarded as a linear space over F. The extension $\mathfrak{R}$ is *finite* if the space is finite dimensional, i.e., there are elements $e_1, \ldots, e_n$ in $\mathfrak{R}$ such that every x in $\mathfrak{R}$ can be written as a unique linear combination of these elements with coefficients in F, namely,

$$x = k_1 e_1 + \ldots + k_n e_n \ (k_1, \ldots, k_n \in F).$$

A set $e_1, \ldots, e_n$ with these properties is a *basis* for $\mathfrak{R}$ over F. In other words, a basis is the largest linearly independent set of elements in $\mathfrak{R}$ with respect to field F.

Field extensions have the following easily proved properties.

1) *Every finite extension $\mathfrak{R}$ of a field F is algebraic.*

For, every basis for a finite dimensional space has the same number of elements. Therefore if $x \in \mathfrak{R}$ and n is the dimension of $\mathfrak{R}$ over F, there are at most n independent powers among $x^0 = e, x, \ldots, x^n$. Hence there exist field elements $a_0, \ldots, a_n$ not all vanishing such that $a_0 + a_1 x + \cdots + a_n x^n = 0$.

2) *If $\mathfrak{R}$ is a finite field extension of F_1 and F_1 is a finite field extension of F_2, then $\mathfrak{R}$ is a finite extension of F_2.*

Let $\{a_1, \ldots, a_m\}$ be a basis for F_1 over F_2 and $\{b_1, \ldots, b_n\}$ a basis for $\mathfrak{R}$ over F_1. Then the elements $a_i b_j$ $(i = 1, \ldots, m; j = 1, \ldots, n)$ comprise a basis for $\mathfrak{R}$ over F_2.

3) *The splitting field $\mathfrak{R}$ of a polynomial $f(x)$ over a field F is finite and hence algebraic.*

Consider the adjunction to F of a single root a of $f(x) = 0$. This yields a finite extension F_1 with basis $e, a, \ldots, a^{r-1}$, where r is the degree of the irreducible polynominal in $F[x]$ of which a is a root. If we now adjoin to F_1 another root b of $f(x)$, we obtain a finite field extension F_2 of F_1, and so on.

Using transfinite induction, we can easily prove that the splitting field $\mathfrak{K}$ of any set of polynominals over a field F is algebraic over F.

4) *If $\mathfrak{K}$ is an algebraic field extension of F_1 and F_1 is an algebraic extension of F_2, then $\mathfrak{K}$ is an algebraic extension of F_2.*

Suppose $a \in \mathfrak{K}$. Then a is the root of some polynominal with coefficients $a_0, \ldots, a_n$ in F_1 which are in turn algebraic over F_2. If we adjoin $a_0, \ldots, a_n$ to F_2, we obtain a finite extension F_2' and the field $F_2'(a)$ is finite over F_2'. Hence $F_2'(a)$ is finite over F_2 and so the element a is algebraic over F_2.

A field $\mathfrak{F}$ is said to be *algebraically closed* if every nonconstant polynominal in the ring $\mathfrak{F}[x]$ has at least one root in $\mathfrak{F}$. A basic theorem in the algebra of complex numbers asserts that the field of complex numbers is algebraically closed.

Existence Theorem. *Every field $\mathfrak{F}$ has an algebraically closed algebraic extension $\mathfrak{K}$.*

Let F denote the set of all irreducible polynomials in the ring $\mathfrak{F}[x]$ of at least second degree. With each f in F, associate in one-to-one fashion an indeterminate y_f and form the polynomial ring $\mathfrak{F}[Y]$ with respect to the set Y of all such indeterminates. It includes in particular the polynomials $f(y_f)$ for each $f(x) \in F$. Let I be the smallest ideal in $\mathfrak{F}[Y]$ containing all $f(y_f)$. Observe that the identity $e \bar{\in} I$. For, assuming to the contrary that it did, we would have

$$e = \varphi_1 f_1(y_{f_1}) + \ldots + \varphi_k f_k(y_{f_k}),$$

where $\varphi_1, \ldots, \varphi_k$ are polynominals in $\mathfrak{F}[Y]$. If we adjoined a root of $f_1, \ldots, f_k$ one by one to $\mathfrak{F}$, we would obtain an extension field of $\mathfrak{F}$. The replacement of the indeterminates $y_{f_1}, \ldots, y_{f_k}$ by the roots of the corresponding polynomials $f_1, \ldots, f_k$ in this extension would lead to the fact that $e = 0$. Thus, $I \neq \mathfrak{F}[Y]$.

By Zorn's lemma (see Section 1.5), I is contained in a maximal ideal I' of the ring $\mathfrak{F}[Y]$ which does not coincide with $\mathfrak{F}[Y]$. By Theorem 3 of Section 4.1, the quotient ring $\bar{\mathfrak{F}} = \mathfrak{F}[Y]/I'$ is a field. Since $I' \neq \mathfrak{F}[Y]$, $\mathfrak{F} \cap I' = \{0\}$ and we may regard $\mathfrak{F}$ as a subset of $\bar{\mathfrak{F}}$. Each polynomial $f(x)$ in F has at least one root in $\bar{\mathfrak{F}}$, for example, $I' + y_f$, since $f(I' + y_f) = I' + f(y_f) = I'$. But the field $\bar{\mathfrak{F}}$ is generated by these roots (in the sense of Section 3.2) and so $\bar{\mathfrak{F}}$ is an algebraic extension of $\mathfrak{F}$.

Thus, every field $\mathfrak{F}$ has an algebraic extension $\overline{\mathfrak{F}}$ in which each irreducible polynominal in the ring $\mathfrak{F}[x]$ has at least one root.

Starting with arbitrary field $\mathfrak{F}$, we can construct an ascending chain of fields

$$\mathfrak{F}_0 = \mathfrak{F} \subset \mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \dots$$

such that each successive field $\mathfrak{F}_{n+1}$ is an algebraic extension of the predecessor $\mathfrak{F}_n$ and every irreducible polynominal in the ring $\mathfrak{F}_n[x]$ has at least one root in the field $\mathfrak{F}_{n+1}$. Let $\mathfrak{K}$ be the union of the chain of fields. By Section 2.3, $\mathfrak{K}$ is a field. Since $\mathfrak{F}_n$ is an algebraic extension of $\mathfrak{F}$ for $n = 1, 2, \dots$, $\mathfrak{K}$ is also algebraic over $\mathfrak{F}$. If $f(x) = a_0 + a_1 x + \dots + a_k x^k$ is a polynomial in the ring $\mathfrak{K}[x]$ irreducible over $\mathfrak{K}$, then its coefficients $a_0, a_1, \dots, a_k$ belong to some field $\mathfrak{F}_n$ and therefore $f(x)$ belongs to the ring $\mathfrak{F}_n[x]$. It is clear that $f(x)$ is irreducible over the field $\mathfrak{F}_n$ and by construction it has at least one root in the field $\mathfrak{F}_{n+1}$ and hence in the field $\mathfrak{K}$. Thus, $\mathfrak{K}$ is an algebraically closed algebraic extension field of $\mathfrak{F}$.

Also valid is the following uniqueness theorem which we shall state without proof.

Uniqueness Theorem. *Every pair of algebraically closed algebraic extensions $\mathfrak{K}$ and $\mathfrak{K}'$ of a field F are equivalent over F or, in other words, there exists an isomorphism of $\mathfrak{K}$ onto $\mathfrak{K}'$, which on F is the identity mapping.*

This implies in particular that *an algebraically closed algebraic extension $\mathfrak{K}$ of F contains (up to equivalence on F) all algebraic extension fields of F.*

Indeed, suppose, F_1 is an arbitrary algebraic extension field of F and $\mathfrak{K}'$ is an algebraically closed extension field of F_1. Since $\mathfrak{K}'$ is an algebraic extension of F, $\mathfrak{K}'$ is equivalent to $\mathfrak{K}$ over F. Hence it follows that F_1 is equivalent to some subfield of $\mathfrak{K}$.

The algebraically closed algebraic extension field of the set of rational numbers is called the *field of algebraic numbers*.

Let F be a given field and let $\mathfrak{F}$ be an extension of it. A finite set of elements $u_1, \dots, u_n$ in $\mathfrak{F}$ is said to be *algebraically independent over F* if for any polynomial $f(x_1, \dots, x_n)$ in $x_1, \dots, x_n$ with coefficients in the field F the following implication is true:

$$f(u_1, \dots, u_n) = 0 \Rightarrow f(x_1, \dots, x_n) = 0.$$

An arbitrary set S of elements in $\mathfrak{F}$ is said to be algebraically independent over F if every finite subset of S is algebraically independent over F. The collection of all sets of elements of $\mathfrak{F}$ algebraically independent over F can be well-ordered by set-theoretic inclusion and if $S_1 \subseteq S_2 \subseteq \dots$ are algebraically independent sets over F, then their union S is also algebraically independent over F. By Zorn's lemma, to every set S of ele-

ments of $\mathfrak{F}$ algebraically independent over F, there exists a maximal set $S_{\max}$ of elements in $\mathfrak{F}$ algebraically independent over F. All maximal sets of elements of $\mathfrak{F}$ algebraically independent over F can be shown to have the exact same cardinality in the same way as one proves that all maximal linearly independent sets of vectors in a linear space have the same cardinality. It is called the *transcendence degree* of F with respect to F.

Let S be an arbitrary set of elements of $\mathfrak{F}$ algebraically independent over F. With each element u in S, associate an indeterminate x and let X be the set of all such indeterminates. The correspondence

$$\frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)} \to \frac{f(u_1, \ldots, u_n)}{g(u_1, \ldots, u_n)} \qquad (f, g \in F[X], x_1, \ldots, x_n \in X)$$

is an isomorphism of the field of polynomial quotients $F(X)$ onto the extension $F(S)$. Therefore the algebraically independent elements over F will behave in the same way as their corresponding indeterminates.

The extension $F(S)$ of F obtained by the adjunction of an arbitrary set of elements algebraically independent over F is said to be a *purely transcendental* extension field of F.

Steinitz' Theorem. *Every extension field $\mathfrak{K}$ of F is an algebraic extension of some purely transcendental extension field F_1 of F.*

To this end, let $F_1 = F(S_{\max})$, where $S_{\max}$ is some maximal set of elements of $\mathfrak{K}$ algebraically independent over F. By definition, F_1 is a purely transcendental extension field of F. If $v \in \mathfrak{K}$, the set of elements $\{v, S_{\max}\}$ is not algebraically independent over F and therefore there is a nontrivial polynomial $f(x_1, \ldots, x_{n+1})$ in the indeterminates $x_1, \ldots, x_{n+1}$ with coefficients in F such that

$$f(v, u_1, \ldots, u_n) = 0$$

for $u_1, \ldots, u_n$ in $S_{\max}$. Writing the polynomial $f(x)$ in decreasing powers of v, we have

$$a_0(u)v^m + a_1(u)v^{m-1} + \cdots + a_m(u) = 0,$$

where $a_i(u) = a_i(u_1, \ldots, u_n) \in F_1$ $(i = 0, 1, \ldots, m)$. Since the elements in $S_{\max}$ are algebraically independent over F, the degree of f with respect to x_1 is nonzero. Thus, the element v is algebraic over F_1, as asserted.

Let $\mathfrak{F}$ be any field and let F be its prime subfield. The transcendence degree of $\mathfrak{F}$ with respect to F is then called simply the *transcendence degree of* $\mathfrak{F}$.

Corollary 1. *All algebraically closed fields of the same characteristic* $\varkappa$ *having given transcendence degree* t *are isomorphic.*

To see this, let $\mathfrak{F}$ and $\mathfrak{F}'$ be algebraically closed fields of characteristic $\varkappa$ and transcendence degree t. They may be considered to be extensions of the same prime field F. By Steinitz' theorem, $\mathfrak{F}$ and $\mathfrak{F}'$ are algebraically closed algebraic extensions of the same purely transcendental extension field $F(S)$ of F, where $|S| = \mathfrak{t}$. By the uniqueness theorem, $\mathfrak{F}$ and $\mathfrak{F}'$ are isomorphic.

Corollary 2. *All algebraically closed fields of fixed characteristic* $\varkappa$ *having the same uncountably infinite cardinality* $\mathfrak{m}$ *are isomorphic.*

Indeed, let $\mathfrak{F}$ be any algebraically closed field of characteristic $\varkappa$ and transcendence degree t having cardinality $\mathfrak{m} > \aleph_0$. Let F be the prime subfield of $\mathfrak{F}$. Choose any maximal set of elements $S_{\max}$ algebraically independent over F and adjoin them to F. Denote the resultant extension by T. Each element v of $\mathfrak{F}$ is the root of a polynomial with coefficients in T. Therefore

$$\mathfrak{m} \leqq |T[x]| \leqq |T| + |T|^2 + |T|^3 + \cdots$$

By assumption, the cardinality of $\mathfrak{F}$ is infinite and uncountable. Hence, the cardinality $|T|$ of T has to be infinite. But then $|T| + |T|^2 + \cdots = |T|$ (see item 3 in the Problems and Complements, Section 2). Thus, $\mathfrak{m} = |T|$. On the other hand, it is not hard to show that $|T| = \mathfrak{t}$. Thus, $\mathfrak{t} = \mathfrak{m}$ and Corollary 2 is proved.

4.3 Alternative Skew Fields. A ring $\mathfrak{R}$ is said to have the *inverse property* if for each nonzero element a in $\mathfrak{R}$ there exists an element a^{-1} such that

$$a^{-1}(ax) = (xa)a^{-1} = x$$

for every x in $\mathfrak{R}$.

Theorem 1. *If a ring* $\mathfrak{R}$ *with identity* e *has the inverse property, then* $\mathfrak{R}$ *is an alternative skew field.*

Suppose $ab = 0$ and $a \neq 0$ $(a, b \in \mathfrak{R})$. Then $a^{-1}(ab) = b = 0$. Similarly, suppose $ab = 0$ and $b \neq 0$. Then $(ab)b^{-1} = a = 0$. Thus $\mathfrak{R}$ has no zero divisors and being a ring with division and with identity, it is a skew field.

Let a and b be any nonzero elements of $\mathfrak{R}$. If we set $x = (ab)^{-1}$, we obtain $x^{-1} = ab$ and $x^{-1}b^{-1} = a$ which implies $xa = x(x^{-1}b^{-1}) = b^{-1}$. In other words, the usual rule for finding the inverse of a product, $(ab)^{-1} = b^{-1}a^{-1}$, holds in $\mathfrak{R}$.

Applying this rule, we can prove the alternative law

$$ab \cdot a = a \cdot ba. \tag{1}$$

Right off, we may suppose that neither a nor b vanishes and that $b \neq a^{-1}$. We have

$$\begin{aligned} a - a(ba) &= (b^{-1} - a) \cdot ba = [a^{-1}b^{-1} \cdot (b^{-1} - a)^{-1}]^{-1} \\ &= \left[\left(e + a^{-1}(b^{-1} - a)\right)(b^{-1} - a)^{-1}\right]^{-1} \\ &= [(b^{-1} - a)^{-1} + a^{-1}]^{-1}. \end{aligned}$$

On the other hand.

$$\begin{aligned} a - (ab)a &= ab \cdot (b^{-1} - a) = [(b^{-1} - a)^{-1} \cdot b^{-1}a^{-1}]^{-1} \\ &= \left[(b^{-1} - a)^{-1}\left(e + (b^{-1} - a)a^{-1}\right)\right]^{-1} \\ &= [(b^{-1} - a)^{-1} + a^{-1}]^{-1}. \end{aligned}$$

Thus $a - a(ba) = a - (ab)a$ and so $a \cdot ba = ab \cdot a$.

From the equalities $(ab \cdot a^{-1})a = a(ba^{-1} \cdot a) = (a \cdot ba^{-1})a$ and the fact that there are no zero divisors, it follows that

$$ab \cdot a^{-1} = a \cdot ba^{-1} \tag{2}$$

identically.

The alternative law $a \cdot ab = a^2b$ is proved as follows. Let $ab \neq 0$ and $b \neq -a^{-1}$. On the basis of (1), we can equate the expressions

$$\begin{aligned} (b + a^{-1})(b^{-1}a) \cdot (b + a^{-1}) &= (a + a^{-1} \cdot b^{-1}a)(b + a^{-1}) \\ &= ab + (a^{-1} \cdot b^{-1}a)b + e + a^{-1}b^{-1}, \end{aligned}$$

and

$$\begin{aligned} (b + a^{-1}) \cdot (b^{-1}a)(b + a^{-1}) &= (b + a^{-1})(b^{-1}a \cdot b + b^{-1}) \\ &= ab + e + a^{-1}(b^{-1}a \cdot b) + a^{-1}b^{-1}. \end{aligned}$$

This yields

$$a^{-1}(b^{-1}a \cdot b) = (a^{-1} \cdot b^{-1}a)b. \tag{3}$$

Replacing a by $(a + b^{-1})^{-1}$, b by a and taking (2) into consideration, we have

$$\begin{aligned} &(a + b^{-1})[a^{-1}(a + b^{-1})^{-1} \cdot a] \\ &\quad = (a + b^{-1})[(a^2 + b^{-1}a)^{-1}a] \\ &\quad = (a + b^{-1})(a + a^{-1}b^{-1}a)^{-1} \\ &\quad = a(a + a^{-1} \cdot b^{-1}a)^{-1} + b^{-1}(a + a^{-1} \cdot b^{-1}a)^{-1} \\ &\quad = [e + (a^{-1} \cdot b^{-1}a)a^{-1}]^{-1} + [ab + (a^{-1} \cdot b^{-1}a)b]^{-1} \\ &\quad = (e + a^{-1}b^{-1})^{-1} + [ab + (a^{-1} \cdot b^{-1}a)b]^{-1} \end{aligned}$$

and

$$\begin{aligned}(a + b^{-1})(a^{-1}(a + b^{-1})^{-1}) \cdot a &= (a + b^{-1})(a^2 + b^{-1}a)^{-1} \cdot a \\ &= [a(a^2 + b^{-1}a)^{-1} + b^{-1}(a^2 + b^{-1}a)^{-1}] \cdot a \\ &= [(a + b^{-1})^{-1} + (a^2 b + b^{-1}a \cdot b)^{-1}]a \\ &= (e + a^{-1}b^{-1})^{-1} + [a^{-1} \cdot a^2 b + a^{-1}(b^{-1}a \cdot b)]^{-1}.\end{aligned}$$

Hence, we obtain

$$ab + (a^{-1} \cdot b^{-1}a)b = a^{-1} \cdot a^2 b + a^{-1}(b^{-1}a \cdot b).$$

By virtue of (3), $ab = a^{-1} \cdot a^2 b$, i.e., $a \cdot ab = a^2 b$.

The converse is also valid.

Theorem. 2 *An alternative ring $\mathfrak{R}$ with division and no zero divisors contains an identity and has the inverse property.*

If we evaluate the associator $[x + y, x + y, z]$ and take into account that it is equal to the zero element in $\mathfrak{R}$, we obtain

$$[x, y, z] = -[y, x, z].$$

Similarly, $[x, y, z] = -[x, z, y]$ and $[x, y, z] = -[z, y, x]$. Thus in an alternative ring the associator changes sign whenever a pair of its arguments are permuted.

If the associators are written out explicitly, it is easy to prove the following identity for any ring:

$$[xy, z, u] - [x, yz, u] + [x, y, zu] = x[y, z, u] + [x, y, z]u. \tag{4}$$

Making use of this identity, we can show that every three elements a, b and x of an alternative ring satisfy

$$[a, ba, x] = a[a, b, x]. \tag{5}$$

Indeed, replacing x and z by a, y by b and u by x in (4), we obtain

$$[ab, a, x] - [a, ba, x] + [a, b, ax] = a[b, a, x]$$

since $[a, b, a] = 0$. This implies

$$[a, ba, x] = a[a, b, x] + [a, b, ax] - [a, ab, x]. \tag{6}$$

Replacing x and y by a, z by b and u by x in (4), we obtain

$$[a^2, b, x] - [a, ab, x] = a[a, b, x],$$

or

$$[a, ab, x] = [a^2, b, x] - a[a, b, x].$$

Substituting this expression for $[a, ab, x]$ in (6), we have

$$[a, ba, x] = 2a[a, b, x] + [a, b, ax] - [a^2, b, x]. \tag{7}$$

Replacing x and y by a, z by x and u by b in (4), we obtain

$$[a^2, x, b] - [a, ax, b] = a[a, x, b],$$

i.e., $[a, b, ax] - [a^2, b, x] = -a[a, b, x]$. Substituting this expression in (7), we finally arrive at (5).

Suppose now $a \in \mathfrak{R}$, $a \neq 0$. By assumption, there exists an element $e \in \mathfrak{R}$ such that $ae = a$. Hence , $aa = ae \cdot a = a \cdot ea$. Cancelling a on the left, we obtain $a = ea$. In view of (5), every element $x \in \mathfrak{R}$ satisfies the equation

$$[a, ea, x] = a[a, e, x].$$

This implies $[a, e, x] = 0$, i.e., $ax = a \cdot ex$ and therefore $x = ex$. From (5) we also find that $[x, ex, a] = x[x, e, a]$ from which $x[x, e, a] = 0$. If $x \neq 0$, we have $xe \cdot a = x \cdot ea = xa$, i.e., $xe = x$. Thus e is the identity in $\mathfrak{R}$.

To each nonzero element $a \in \mathfrak{R}$, there is by assumption an element a^{-1} such that $aa^{-1} = e$. We also have

$$ae = ea = aa^{-1} \cdot a = a \cdot a^{-1}a,$$

and this implies $a^{-1}a = e$. Further, by (5),

$$0 = [a^{-1}, aa^{-1}, x] = a^{-1}[a^{-1}, a, x],$$

and so $[a^{-1}, a, x] = 0$, i.e., $a^{-1}(ax) = x$. Since $[x, a, a^{-1}] = -[a^{-1}, a, x] = 0$, we also have $(xa)a^{-1} = x$. Therefore, $\mathfrak{R}$ has the inverse property and so Theorem 2 is proved.

An example of an alternative nonassociative skew field will be considered in the next subsection.

4.4 Linear Algebras. Let $\mathfrak{F}$ be a field whose elements are $\alpha, \beta, \gamma, \ldots$ and whose identity element is ε. A *linear algebra* over $\mathfrak{F}$ is a ring $\mathfrak{L}$ in which for each α in $\mathfrak{F}$ there is defined a unary operation ω_α such that the following axioms hold in $\mathfrak{L}$:

(V) $\omega_\alpha(x + y) = \omega_\alpha(x) + \omega_\alpha(y)$,

(VI) $\omega_{\alpha+\beta}(x) = \omega_\alpha(x) + \omega_\beta(x)$,

(VII) $\omega_{\alpha\beta}(x) = \omega_\alpha(\omega_\beta(x))$,

(VIII) $\omega_\varepsilon(x) = x$,

(IX) $\omega_\alpha(xy) = (\omega_\alpha(x))y = x(\omega_\alpha(y))$ $(x, y \in \mathfrak{L})$.

Thus, a linear algebra over a field $\mathfrak{F}$ is a universal algebra with basic operations $x + y$, $-x$, xy, and $\omega_\alpha(x)$ $(\alpha \in \mathfrak{F})$ related to each other by Axioms (I)—(IV) of Section 4.1 and the above Axioms (V)—(IX). The algebra $\mathfrak{L}$ over $\mathfrak{F}$ is a linear space with respect to the operations $x + y$, $-x$ and $\omega_\alpha(x)$ $(\alpha \in \mathfrak{F})$ in which

$$\alpha x = \omega_\alpha(x) \qquad (x \in \mathfrak{L},\ \alpha \in \mathfrak{F}).$$

If the linear space is finite dimensional, $\mathfrak{L}$ is said to be *finite* over $\mathfrak{F}$. Let $a_1, \ldots, a_n$ be a basis for a finite algebra $\mathfrak{L}$ over $\mathfrak{F}$. Being elements of $\mathfrak{L}$, the products $a_i a_j$ must be expressible as linear combinations of the basis. Therefore the following relations are satisfied in $\mathfrak{L}$:

$$a_i a_j = \gamma_{ij1} a_1 + \gamma_{ij2} a_2 + \cdots + \gamma_{ijn} a_n \qquad (i, j = 1, \ldots, n). \tag{1}$$

The n^3 coefficients γ_{ijk} are called the *constants of structure* of the algebra $\mathfrak{L}$ in the basis $a_1, \ldots, a_n$. They depend on the choice of basis but they completely determine the multiplication in $\mathfrak{L}$.

Suppose n^3 elements γ_{ijk} $(i, j, k = 1, \ldots, n)$ of a field are given. Choose arbitrary indeterminates $e_1, \ldots, e_n$ and denote by $\mathfrak{L}$ the linear space of all linear combinations $\alpha_1 e_1 + \cdots + \alpha_n e_n$ of $e_1, \ldots, e_n$ with coefficients in $\mathfrak{F}$. Define a multiplication for the elements of $\mathfrak{L}$ by the formula

$$\sum \alpha_i e_i \cdot \sum \beta_j e_j = \sum_{i,j,k} \alpha_i \beta_j \gamma_{ijk} e_k.$$

This converts $\mathfrak{L}$ into a linear algebra over $\mathfrak{F}$ with constants of structure γ_{ijk}. Thus any set of n^3 elements γ_{ijk} may be viewed as the structure constants of some linear algebra.

The system of relations (1) is referred to as the *multiplication table for basis* $a_1, \ldots, a_n$.

If the algebra $\mathfrak{L}$ has an identity e, its elements αe $(\alpha \in \mathfrak{F})$ form a subalgebra isomorphic to $\mathfrak{F}$. Thus identifying αe with α for each α in $\mathfrak{F}$, we can embed $\mathfrak{F}$ in $\mathfrak{L}$ as a subalgebra.

Let us consider several examples of linear algebras. First of all, we have the field of complex numbers which is an algebra over the reals for which the numbers 1 and i form a basis with multiplication table,

$$1^2 = 1, \quad 1 \cdot i = i \cdot 1 = i, \quad i^2 = -1.$$

In the middle of the last century, Hamilton concocted an algebra of dimension 4 over the field of real numbers. It is an associative but noncommutative field. Its elements are called *quaternions*.

A basis for the quaternion algebra $\mathfrak{Q}$ is 1, i, j and k, the first element being the identity of the algebra $\mathfrak{Q}$ and identified with the real number 1.

The remaining basis elements are multiplied according to the following table:

$$\begin{aligned} i^2 &= j^2 = k^2 = -1, \\ ij &= -ji = k, \\ jk &= -kj = i, \\ ki &= -ik = j. \end{aligned}$$

The table can be remembered easily with the help of the diagram indicated in Fig. 1. Thus, each quaternion is uniquely expressible as a

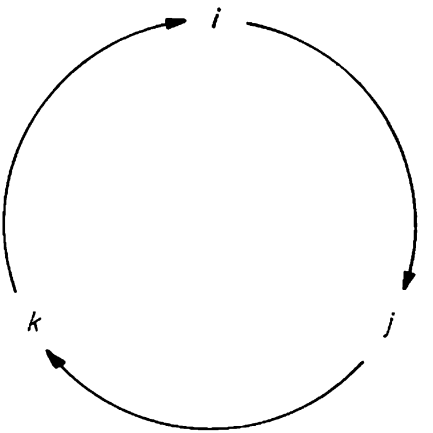

Fig. 1

linear combination $\alpha + \beta i + \gamma j + \delta k$ of the basis 1, i, j and k with real coefficients α, β, γ and δ. Quaternions are multiplied using the distributive law and the multiplication table for the basis elements.

The multiplication table clearly shows that $\mathfrak{Q}$ is an associative but noncommutative algebra: for any basis elements x, y and z, we have $(xy)z = x(yz)$.

The nonnegative number $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is called the *norm* of the quaternion $q = \alpha + \beta i + \gamma j + \delta k$ and is denoted by $N(q)$.

The *conjugate* of $q = \alpha + \beta i + \gamma j + \delta k$ is the quaternion $\alpha - \beta i - \gamma j - \delta k$ and is denoted by $\bar{q}$.

Direct computation shows that

$$\overline{q_1 + q_2} = \bar{q}_1 + \bar{q}_2, \quad \overline{q_1 q_2} = \bar{q}_2 \bar{q}_1, \quad \overline{\alpha q} = \alpha \bar{q},$$
$$q\bar{q} = \bar{q}q = N(q), \quad N(q_1 q_2) = N(q_1) N(q_2),$$

where q, q_1 and q_2 belong to $\mathfrak{Q}$ and α is in $\mathfrak{F}$.

If $q \in \mathfrak{Q}$, $q \neq 0$, then defining

$$q^{-1} = \frac{1}{N(q)}\bar{q},$$

we have

$$q^{-1}q = qq^{-1} = 1.$$

Thus each nonzero element q of the multiplicative semigroup of $\mathfrak{Q}$ is invertible and *the algebra $\mathfrak{Q}$ is a skew field.*

Quaternions are not as widely used as complex numbers but they are nevertheless used both in mathematics and in mechanics. Let us consider one such algebraic application to the problem of composing real quadratic forms, i.e., the problem of representing the product of two sums of squares of quantities as a sum of squares of bilinear forms of these quantities.

The simplest rule for composing quadratic forms (in two variables) is furnished by the complex numbers. If $u = x_1 + x_2 i$ and $v = y_1 + y_2 i$, then the formula $N(u)N(v) = N(uv)$ can be expressed as

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 - x_2y_2)^2 + (x_1y_2 + x_2y_1)^2.$$

Taking the quaternions

$$u = x_1 + x_2 i + x_3 j + x_4 k, \quad v = y_1 + y_2 i + y_3 j + y_4 k,$$

we can concoct the following rule for composing quadratic forms in four variables:

$$\begin{aligned}&(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2)\\ &= (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)^2\\ &\quad + (x_1y_3 + x_3y_1 + x_4y_2 - x_2y_4)^2 + (x_1y_4 + x_4y_1 + x_2y_3 - x_3y_2)^2,.\end{aligned}$$

The search for a way of composing forms in eight variables led Cayley to the discovery in the 1850's of a special algebra of dimension 8 whose elements are now known as *Cayley numbers.*

Let K be the set of ordered pairs (q_1, q_2) of quaternions q_1 and q_2. Define addition and multiplication by a real number α in K by the formulas

$$\begin{aligned}(q_1, q_2) + (r_1, r_2) &= (q_1 + r_1, q_2 + r_2),\\ \alpha(q_1, q_2) &= (\alpha q_1, \alpha q_2).\end{aligned}$$

Under these operations, K becomes a linear space over the field of real numbers (i.e., Axioms (I)—(III) of Section 4.1 and Axioms (V)—(VIII) of the current subsection hold in K).

A multiplication is definable in the space by

$$(q_1, q_2)(r_1, r_2) = (q_1 r_1 - \bar{r}_2 q_2, r_2 q_1 + q_2 \bar{r}_1), \tag{2}$$

where $\bar{r}$ is the conjugate quaternion. Axiom (IX) is trivial. The distributive laws for multiplication over addition are also easy to verify. Thus $\mathfrak{K} = \langle K, +, \cdot, \{\omega_\alpha\}\rangle$ is a linear algebra over the field of real numbers. It is known as the *Cayley algebra.*

The pairs of the form $(q, 0)$ comprise a subalgebra isomorphic to the quaternion algebra, the isomorphic mapping being $(q, 0) \to q$. Therefore a Cayley number of the form $(q, 0)$ can be identified with the quaternion q and so by that token, the quaternion algebra $\mathfrak{Q}$ is embeddable in the Cayley algebra $\mathfrak{K}$ as a subalgebra.

Observe that

$$(q_1, q_2) = (q_1, 0) + (q_2, 0)(0, 1).$$

If we now set $e = (0, 1)$, we obtain for each Cayley number an equivalent notation

$$q_1 + q_2 e, \tag{3}$$

where q_1 and q_2 are quaternions.

If

$$q_1 = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k,$$

and

$$q_2 = \alpha_4 + \alpha_5 i + \alpha_6 j + \alpha_7 k,$$

we can write the Cayley number $q_1 + q_2 e$ in the unique form

$$\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k + \alpha_4 e + \alpha_5 ie + \alpha_6 je + \alpha_7 ke, \tag{4}$$

where $\alpha_0, \ldots, \alpha_7$ are real numbers. Hence the elements $1, i, j, k, e, ie, je$, and ke comprise a basis for the Cayley algebra over the field of real numbers.

In practice, it is more convenient to write the Cayley numbers in form (3) rather than (4) and to make use of the relations

$$\left.\begin{aligned} e^2 = -1, \quad eq = \bar{q}e, \quad p \cdot qe = q\bar{p} \cdot e, \\ pe \cdot q = p\bar{q} \cdot e, \quad pe \cdot qe = -\bar{q}p, \end{aligned}\right\} \tag{5}$$

when carrying out computations, where p and q are quaternions. These relations are immediate consequences of formula (2).

Relations (5) yield a multiplication table for the basis $1, i, j, k, e, ie, je$ and ke. In particular, we have

$$i \cdot je = ji \cdot e = -ke, \quad ij \cdot e = ke.$$

Thus, a Cayley algebra is nonassociative.

The *conjugate* of Cayley number $u = r + qe$ is

$$\bar{u} = \bar{r} - qe.$$

By use of relations (5), it is easy to derive the following formulas:

$$\overline{\alpha u + \beta v} = \alpha \bar{u} + \beta \bar{v}, \quad \overline{uv} = \bar{v}\bar{u},$$
$$u\bar{u} = \bar{u}u = r\bar{r} + q\bar{q},$$

in which u and v are Cayley numbers, r and q are quaternions and α and β are real numbers.

The nonnegative real number

$$u\bar{u} = r\bar{r} + q\bar{q}$$

is called the *norm* of the Cayley number u and is denoted by $N(u)$. Since

$$N(r + qe) = N(r) + N(q),$$

it follows that $N(u) = 0$ if and only if $u = 0$.

Direct computation shows that every pair of Cayley numbers u and v satisfies the relation

$$\bar{u}(uv) = (vu)\bar{u} = N(u)v.$$

Therefore each nonzero Cayley number u has a well-defined inverse

$$u^{-1} = \frac{1}{N(u)}\bar{u}.$$

By Theorem 1 of Section 4.3, we conclude that *the Cayley algebra is an alternative (nonassociative) skew field.*

By making use of the Cayley numbers, one can easily manufacture a composition rule for real quadratic forms in eight variables. However, at the end of the last century, A. Hurwitz showed that there is no such composition rule for quadratic forms in a greater number of variables. The following theorem also holds.

Theorem. *Every finite dimensional algebra over the field of real numbers having an identity and the inverse property is isomorphic to one of the following*: 1) *the field of real numbers*, 2) *the field of complex numbers*, 3) *the quaternion algebra*, or 4) *the Cayley algebra.*

This theorem was already known for associative-commutative algebras back in the first half of the 19th century. The case of associative non-

commutative algebras was treated by Frobenius. The proof of the theorem in its general form stems from the results of Moufang [52], Zorn [77], [78] and Linnik [33]. Since no use will be made of the theorem in the sequel, we shall not stop to prove it.

Problems and Complements

1. Every nontrivial associative ring $\mathfrak{R}$ with division is a division ring.
2. Every algebraically closed field is infinite.
3. The automorphism group of the field of complex numbers has cardinality $2^{2^{\aleph_0}}$ (Soundararajan [67]).
4. A ring $\mathfrak{R}$ is alternative if and only if all of its subrings generated by two elements are associative (see Širšov [65], pp. 7—8, as well as Kuroš [30]).

5. Lattices (Structures)

5.1 Lattices. We recall that an element m of a partially ordered set M is an *upper bound* for a proper subset N of M if each element of N is comparable with m and does not exceed m. If there is a smallest among the upper bounds of N, it is called the *least upper bound* or *supremum* of N. A *greatest lower bound* or *infimum* of N is defined in a similar way.

A partially ordered set M is called an *upper semilattice* if each pair of iis elements has a supremum and a *lower semilattice* if each pair of its elements has an infimum. A set M is called a *lattice* or is said to *have the lattice property* if each pair of its elements has a supremum and an infimum.

For example, the family of all subsets of a given set A is a lattice under inclusion: the supremum of a pair of subsets A_1 and A_2 is their union and the infimum their intersection. Every linearly ordered set is also a lattice. The supremum of a pair of elements is the larger of the two and the infimum, the smaller of the two.

Let $a \vee b$ be the supremum of elements a and b and let $a \wedge b$ be their infimum. It is easy to see that the following formulas are valid:

(C_1) $a \vee a = a, \quad a \wedge a = a,$

(C_2) $a \vee b = b \vee a, \quad a \wedge b = b \wedge a,$

(C_3) $a \vee (b \vee c) = (a \vee b) \vee c, \quad (a \wedge b) \wedge c = a \wedge (b \wedge c),$

(C_4) $a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$

commutative algebras was treated by Frobenius. The proof of the theorem in its general form stems from the results of Moufang [52], Zorn [77], [78] and Linnik [33]. Since no use will be made of the theorem in the sequel, we shall not stop to prove it.

Problems and Complements

1. Every nontrivial associative ring $\mathfrak{R}$ with division is a division ring.
2. Every algebraically closed field is infinite.
3. The automorphism group of the field of complex numbers has cardinality $2^{2^{\aleph_0}}$ (Soundararajan [67]).
4. A ring $\mathfrak{R}$ is alternative if and only if all of its subrings generated by two elements are associative (see Širšov [65], pp. 7—8, as well as Kuroš [30]).

5. Lattices (Structures)

5.1 Lattices. We recall that an element m of a partially ordered set M is an *upper bound* for a proper subset N of M if each element of N is comparable with m and does not exceed m. If there is a smallest among the upper bounds of N, it is called the *least upper bound* or *supremum* of N. A *greatest lower bound* or *infimum* of N is defined in a similar way.

A partially ordered set M is called an *upper semilattice* if each pair of iis elements has a supremum and a *lower semilattice* if each pair of its elements has an infimum. A set M is called a *lattice* or is said to *have the lattice property* if each pair of its elements has a supremum and an infimum.

For example, the family of all subsets of a given set A is a lattice under inclusion: the supremum of a pair of subsets A_1 and A_2 is their union and the infimum their intersection. Every linearly ordered set is also a lattice. The supremum of a pair of elements is the larger of the two and the infimum, the smaller of the two.

Let $a \vee b$ be the supremum of elements a and b and let $a \wedge b$ be their infimum. It is easy to see that the following formulas are valid:

(C_1) $a \vee a = a, \quad a \wedge a = a,$

(C_2) $a \vee b = b \vee a, \quad a \wedge b = b \wedge a,$

(C_3) $a \vee (b \vee c) = (a \vee b) \vee c, \quad (a \wedge b) \wedge c = a \wedge (b \wedge c),$

(C_4) $a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$

They are called respectively the idempotent laws, the commutative laws, the associative laws and the absorption laws. The distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

need not hold in a lattice.

Laws (C_1)—(C_4) give rise to the following definitions.

An *additive* (*multiplicative*) *semilattice* is a commutative idempotent semigroup $\mathfrak{A} = \langle A, + \rangle$ ($\mathfrak{A} = \langle A, \cdot \rangle$).

An algebra $\langle A, +, \cdot \rangle$ of type $\langle 2, 2 \rangle$ is called an *abstract lattice* if its basic operations are governed by the axioms

(L_1) $a + a = a, \quad aa = a,$

(L_2) $a + b = b + a, \quad ab = ba,$

(L_3) $(a + b) + c = a + (b + c), \quad (ab)c = a(bc),$

(L_4) $a(a + b) = a, \quad a + ab = a$

for all a and b in A.

If A is an upper semilattice, we can convert A into an additive semilattice by defining an addition in A by the formula

$$a + b = a \vee b.$$

The relations $a \leqq b$ and $a + b = b$ are equivalent in the new A.

Conversely, suppose $\langle A, + \rangle$ is an arbitrary additive semilattice. Defining $a \leqq b$ if $a + b = b$, we have $a \leqq a$ since $a + a = a$. If $a \leqq b$ and $b \leqq a$, then $a + b = b$ and $b + a = a$. And since addition is commutative, we obtain $a = b$. Finally if $a \leqq b$ and $b \leqq c$, then $a + b = b$ and $b + c = c$. Applying the associative law, we obtain $a + c = a + (b + c) = b + c = c$, i.e., $a \leqq c$. Hence, A is a partially ordered set with respect to the relation $\leqq$. It is easy to show that the supremum of a and b in A is $a + b$.

A similar connection exists between lower semilattices and multiplicative semilattices as well as between lattices and abstract lattices. Indeed, if A is a lattice, we can convert A into an abstract lattice by setting $a + b = a \vee b$ and $ab = a \wedge b$. Conversely, let A be an abstract lattice. Introduce a relation $\leqq$ in A by defining $a \leqq b$ if $a + b = b$. From the above correspondence between upper semilattices and additive semilattices, it follows that the relation $\leqq$ makes A an upper semilattice in which $a + b = a \vee b$. But the equations $a + b = b$ and $ab = a$ are equivalent in every abstract lattice since $a + b = b$ implies $ab = a(a + b) = a$ and $ab = a$ implies $a + b = ab + b = b + ba = b$.

Therefore the same relation $\leqq$ also makes A a lower semilattice with $ab = a \wedge b$.

Hence, every abstract lattice may be viewed as a set having the lattice property.

The above discussion shows that the assignment of one operation in an abstract lattice completely determines the second one. For example, having an addition, we can introduce the relation $\leqq$. Then with the help

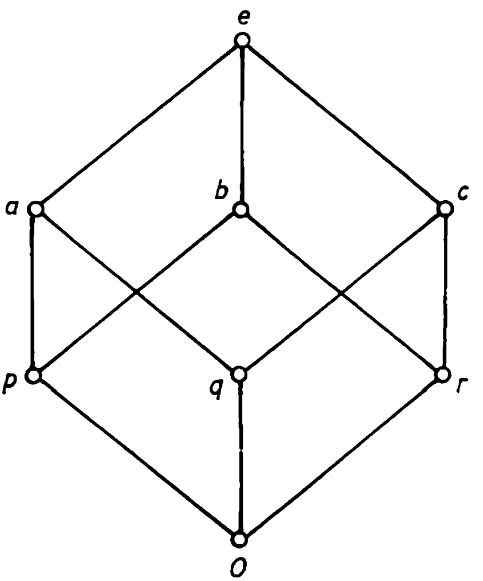

Diagram 1

of it, we can make the product ab the infimum of the pair a and b. But this still does not give us the right to regard addition as the sole basic operation since multiplication cannot be expressed explicitly in terms of it.

The structure of finite partially ordered sets is sometimes given with the help of diagrams. To this end, the elements of a set are represented by points located at different levels, and immediately larger elements are joined to immediately smaller elements by descending lines. An element a then turns out to be larger than b if and only if we can get from a to b in the diagram along lines going downward. For example, Diagram 1 depicts all of the subsets of a three-element set. Element p is less than e but is not comparable with c.

The elements 0 and 1 of an abstract lattice $\mathfrak{A}$ are called the *zero* and the *unit* if for any $a \in \mathfrak{A}$, we have

$$0 + a = a, \quad 1 \cdot a = a.$$

This means that 0 and 1 are respectively the smallest and largest elements of $\mathfrak{A}$. A finite abstract lattice A with elements $a_1, \ldots, a_n$ necessarily has a zero element and unit element, namely,

$$a_1 a_2 \ldots a_n = 0, \quad a_1 + a_2 + \cdots + a_n = 1.$$

Infinite abstract lattices need not have a zero or unit element either separately or simultaneously.

Every finite subset of a lattice has a supremum and an infimum. A partially ordered set is said to be a *complete lattice* if not only every finite but also every infinite subset has a supremum and infimum. For example, the family of all subsets of a set A is a complete lattice. The supremum of any number of subsets is their union and the infimum, their intersection.

If a partially ordered set M has a greatest and smallest element and if every subset of M has an infimum (supremum), then every subset also has a supremum (infimum), or in other words, M is a complete lattice.

To show this, observe that the set P of upper bounds of a subset N of M is nonempty since P necessarily contains the greatest element of M. By assumption, P has an infimum and the infimum is the required supremum for N.

In particular, *the family of all subsystems of any algebraic system $\mathfrak{A}$ (including the empty subsystem) is a complete lattice with respect to inclusion* since any number of subsystems has an infimum equal to their intersection.

The algebraization of the notion of complete lattice necessitates considering operations on an arbitrary set of arguments. Namely, a set M is said to be a *complete abstract lattice* if to each mapping $p \to x_p$ of a subset P of M into M, there correspond uniquely determined elements $\sum\limits_{p \in P} x_p$ and $\prod\limits_{p \in P} x_p$ having the following properties:

1. $\sum\limits_{p \in P} x_p = \prod\limits_{p \in P} x_p = x$ if $x_p = x$ for all p in P.

2. If P is the union of pairwise disjoint subsets P_i, then

$$\sum_i \sum_{p \in P_i} x_p = \sum_{p \in P} x_p, \quad \prod_i \prod_{p \in P_i} x_p = \prod_{p \in P} x_p.$$

3. $a + ab = a$ and $a(a + b) = a$.

Property 2 combines the associative and commutative laws. If the relation

$$a \leqq b \Leftrightarrow a + b = b$$

is introduced in a complete abstract lattice, it is easy to show that it will be a complete lattice in which

$$\sum_{p \in P} x_p = \bigvee_{p \in P} x_p, \quad \prod_{p \in P} x_p = \bigwedge_{p \in P} x_p.$$

The converse is obvious.

The notion of complete abstract lattice may be defined without the use of infinite sums and products. Namely, an abstract lattice is complete if it is a complete lattice with respect to the natural ordering of elements.

In the following, no distinction will be made between lattices and abstract lattices.

5.2 Modular and Distributive Lattices. Boolean Algebras. A lattice $\mathfrak{A}$ is called *modular* (or *Dedekind*)*) if its elements satisfy the modular law

$$a(ab + c) = ab + ac, \tag{1}$$

and *distributive* if its elements satisfy the distributive law

$$a(b + c) = ab + ac. \tag{2}$$

An example of a distributive lattice is the lattice of all subsets of a set. In this lattice, sum and product are the same as union and intersection, respectively, and these operations are distributive.

Every distributive lattice is modular since from (2) we have

$$a(ab + c) = a \cdot ab + ac = ab + ac.$$

The converse is false. The class of modular lattices is broader than the class of distributive lattices.

The importance of modular lattices is apparent from the following fact first noticed by Dedekind: *the lattice of normal subgroups of a group is modular.*

In the lattice of normal subgroups, the infimum of two normal subgroups H_1 and H_2 is their intersection and the supremum, their product

*) Dedekind, R., Über die von drei Moduln erzeugte Dualgruppe, Math. Ann. **53** 371–403 (1900).

H_1H_2 comprising elements of the form h_1h_2 with $h_1 \in H_1$ and $h_2 \in H_2$. Therefore the modular law for normal subgroups reduces to

$$H_1 \cap [(H_1 \cap H_2)H_3] = (H_1 \cap H_2)(H_1 \cap H_3).$$

To prove this, let x be an element of the left-hand side. Then $x \in H_1$ and $x = hh_3$, where $h \in H_1 \cap H_2$ and $h_3 \in H_3$. This implies $h_3 = h^{-1}x$

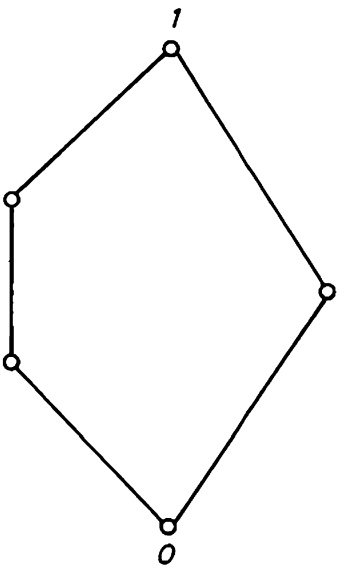

Diagram 2

$\in H_1$ and so $h_3 \in H_1 \cap H_3$. Hence, $hh_3 \in (H_1 \cap H_2)(H_1 \cap H_3)$. Conversely, if x belongs to the right-hand side, then $x = h_2h_3$ where $h_2 \in H_1 \cap H_2$ and $h_3 \in H_1 \cap H_3$. Thus $x \in H_1$ and so x is a member of the left-hand side.

Diagram 2 depicts a nonmodular lattice with the fewest number of elements. It is easy to prove that every nonmodular lattice contains such a sublattice.

The modular law (1) implies the identity

$$(a + b)(a + c) = a + (a + b)c. \tag{3}$$

For,

$$\begin{aligned}(a + b)(a + c) &= (a + b)[(a + b)a + c] \\ &= (a + b)a + (a + b)c = a + (a + b)c.\end{aligned}$$

Conversely, the modular law is impled by (3).

Similarly, the distributive law (2) is equivalent to the law

$$(a + b)(a + c) = a + bc.$$

A lattice $\mathfrak{A}$ is said to be *complemented* if it has a 0 and 1 and each element a has a *complement* a' satisfying the conditions

$$a + a' = 1, \quad aa' = 0.$$

Each element a_i in the lattice of Diagram 3 has a complement and serving as complement is any $a_j \neq a_i$. Therefore general lattices and

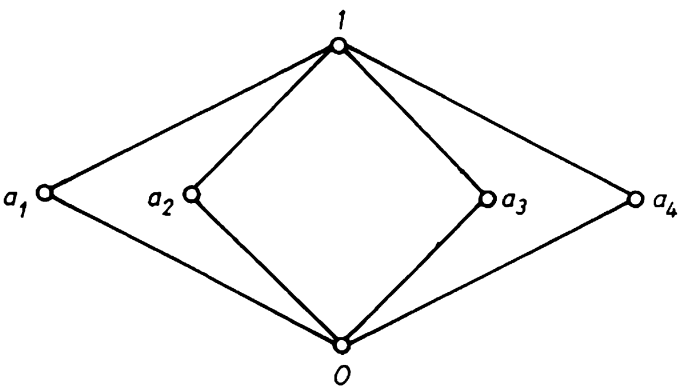

Diagram 3

even modular lattices do not have uniquely determined complements. The matter changes in the case of distributive lattices.

In a distributive lattice with 0 *and* 1, *each element has at most one complement and if a and b have complements a' and b', then*

$$(a')' = a, \quad (a + b)' = a'b', \qquad (4)$$
$$(ab)' = a' + b'.$$

For, if a' and a'' are complements of a, then multiplying the equation $a + a' = 1$ by a'' and applying the distributive law and relations $aa'' = 0$ and $1a'' = a''$, we obtain $a'a'' = a''$. Interchanging the roles of a' and a'', we have $a''a' = a'$. Hence $a' = a''$ and uniqueness is established. We further have

$$(a + b)a'b' = aa'b + a'bb' = 0,$$
$$a + b + a'b' = a(b + b') + (a + a')b + a'b'$$
$$= ab + ab' + a'b + a'b'$$
$$= (a + a')(b + b') = 1.$$

Therefore, $a'b' = (a + b)'$. The other equations are proved in the same way.

A complemented distributive lattice is called a *Boolean lattice*. It is natural to regard complementation in a Boolean lattice as a special unary operation related to addition and multiplication by formulas (4).

An algebra $\langle A, +, \cdot, ' \rangle$ of type $\langle 2, 2, 1 \rangle$ is called a *Boolean algebra* if its basic operations are subject to the set of axioms:

(B_1) $a + a = a, \quad aa = a,$

(B_2) $a + b = b + a, \quad ab = ba,$

(B_3) $a + (b + c) = (a + b) + c, \quad (ab)c = a(bc),$

(B_4) $a(b + c) = ab + ac, \quad a + bc = (a + b)(a + c),$

(B_5) $(a')' = a,$

(B_6) $(a + b)' = a'b'$ and $(ab)' = a' + b',$

(B_7) $(a + a')b = b, \quad aa' + b = b.$

It is easy to see that either only the first laws in these axioms need be retained or only the second ones since the first ones are implied by the second ones and vice versa. Axioms (B_1)–(B_7) yield the absorption laws

$$a(a + b) = a, \quad a + ab = a.$$

Thus, a Boolean algebra is a lattice with respect to addition and multiplication. (B_7) shows that aa' is the least element and $a + a'$ the greatest element of the lattice and hence a' is the complement of a.

Thus, considered only with respect to addition and multiplication, each Boolean algebra is a complemented distributive lattice. Each complemented distributive lattice may in turn be made a Boolean algebra by assuming complementation to be a third basic operation.

The family of all subsets of a set is a Boolean algebra if addition and multiplication are considered to be union and intersection and complementation is complementation of sets. The following converse also holds for finite Boolean algebras.

Stone's Theorem. *Each finite Boolean algebra* $\mathfrak{A}$ *is isomorphic to the algebra of subsets of the set of minimal nonzero elements of* $\mathfrak{A}$.

Since $\mathfrak{A}$ is a finite algebra, there necessarily have to be minimal elements among its nonzero elements. Let M denote the set of them. Set up the following correspondence. With the empty subset of M, associate the zero of $\mathfrak{A}$ and with each subset $\{m_1, \ldots, m_k\}$ of M associate the sum of these elements in $\mathfrak{A}$. It remains to show that the mapping φ

of the algebra of subsets of M into $\mathfrak{A}$ so defined is an isomorphism. Let us show first of all that each nonzero element in $\mathfrak{A}$ is representable as a sum of elements in M. This will show that φ is a mapping *onto* $\mathfrak{A}$. Let a be a nonzero element of $\mathfrak{A}$. If $a \in M$, then all is proved. But if $a \notin M$, there is a b such that $a > b > 0$. Now $a = a(b + b') = ab + ab'$ and $a > ab$ and $a > ab'$. If ab and ab' are minimal elements, then once again all is proved. But if not, we can repeat the same argument on them. Since $\mathfrak{A}$ is finite, the process has to terminate and we obtain the required expansion of a as a sum of minimal elements. All that is left is the one-to-oneness of the mapping φ, i.e., that the representation of a as a sum of distinct nonzero minimal elements is unique. Suppose

$$a = a_1 + \cdots + a_s = b_1 + \cdots + b_t$$

are two such representations. Multiplying both sides by a_i, we obtain $a_i = a_i b_1 + \cdots + a_i b_t$. The products $a_i b_j$ are nonvanishing if and only if $a_i = b_j$. Therefore each term in the first sum occurs in the second and conversely, each term in the second occurs in the first, as required. The stability of addition, multiplication and complementation under the mapping φ is obvious.

For general infinite Boolean algebras, it may be asserted that each infinite Boolean algebra is isomorphic to a subalgebra of the algebra of subsets of some infinite set.

Chapter III

First and Second-Order Languages

6. Syntax and Semantics

6.1 Terms. It has already been stated (in Section 2.2) that the theory of algebraic systems is primarily the study of the properties of systems that remain invariant under isomorphism. One of the languages in which the invariant properties of a system are conveniently expressed is the so-called second-order language. It is constructed in the following way.

By an *alphabet*, we mean any collection of pairwise distinct symbols. We are supposedly able to consider finite sequences or *strings* of letters each "reproducing" a particular symbol of the alphabet. Such strings of letters are *words*. No distinction is to be made between the alphabet symbols and the letters reproducing them. For example, suppose an alphabet consists of the symbols $0, 1, +, =, a$ and b. The strings

$$aa,\ abab,\ a + = 1,\ 0 + 1 = 1,\ +1 = ++a$$

are words in this alphabet. The *length* of a word is the number of letters in it. Two words are (schematically) equal if they are of the same length and their corresponding letters are the same. For example, the words $0 + 1$ and $1 + 0$ are different since their first letters are different. The collection of all words formed from a given alphabet is called a *formal language*. In order to talk about the properties of words in a formal language, we shall employ certain symbols that do not belong to its alphabet. They belong to the *metalanguage* used in discussing the object language and are called *metasymbols*. For example, in phrases such as "consider words A and B of the formal language", the letters A and B are not in the alphabet of the formal language but are metasymbols.

Suppose now that A and B are words in a formal language. If we write down the letters of A and then concatenate (juxtapose) the letters of B, we obtain a new word called the *composition* of A and B. Composition of words will be denoted by AB. It is clearly a special kind of binary operation on the collection of words in a formal language. A dot is sometimes used to denote it and one speaks of word multiplication. The outcome of a multiplication depends of course on the order of

factors. For example,

$$aba \cdot ba = ababa, \quad ba \cdot aba = baaba.$$

However, composition of words is obviously associative.

Apart from the ordinary words in a formal language, we sometimes introduce into consideration an additional special symbol Λ, called the *empty word*. By definition,

$$\Lambda \cdot A = A \cdot \Lambda = A.$$

The length of the empty word is taken to be zero.

Segments of a word are called *subwords*. A word A_1 is clearly a subword of A if and only if there are words X and Y such that

$$A = XA_1Y. \tag{1}$$

A word A may have several segments equal to a given A_1. In that case, equation (1) has several solutions X and Y. The first segment of A equal to A_1 is called the *first embedment* of A_1 in A, the second segment of A equal to A_1 is the *second embedment* of A_1 in A and so on. If the pairs X_i, Y_i $(i = 1, \ldots, n)$ are all solutions of equation (1) and if

$$l(X_1) < l(X_2) < \cdots < l(X_n),$$

where $l(X_i)$ is the length of word X_i, then X_i, Y_i determines the i-th embedment of A_1 in A. The solution X, Y in which either X or Y is Λ is not excluded. If $X_1 = \Lambda$, then A_1 is said to be the initial segment or *initial subword* of A. Segments of length one are simply the letters in a word. In accordance with our definitions, the empty word Λ has $m + 1$ embedments in a word of length m, a fact discernible from the decompositions

$$\Lambda \cdot a_1 \ldots a_m = a_1 \cdot \Lambda \cdot a_2 \ldots a_m = \cdots = a_1 a_2 \ldots a_m \cdot \Lambda.$$

Let A_1 be a subword of A and let B be a given word. Consider the decompositon $A = XA_1Y$ corresponding to the i-th embedment of A_1 in A. *The result of substituting* word B in A for the i-th embedment of A_1 is the word XBY. It is symbolized by $\mathrm{Sb}_i(A; A_1, B)$, where Sb_i stands for the substitution operator. For example,

$$\mathrm{Sb}_2(abac; a, c) = abcc, \quad \mathrm{Sb}_1(ab; \Lambda, c) = cab,$$
$$\mathrm{Sb}_1(cab; a, \Lambda) = cb.$$

At this point, we shall temporarily interrupt our abstract study of formal languages and we shall begin to look at the structure of the

specific second-order language mentioned above. Its alphabet consists of the *logical* symbols

& (for conjunction, read AND),
$\vee$ (for disjunction, read OR),
$\neg$ (for negation, read NOT),
$\rightarrow$ (for implication, read IMPLIES),
$\forall$ (for universal quantification, read FOR EVERY),
$\exists$ (for existential quantification, read THERE EXISTS),
$=$ (for equality, read EQUALS)

plus the symbols

$x, F, P, \Delta,$
((left parenthesis),
) (right parenthesis),
, (comma).

Words of the form

$$(x\Delta\cdots\Delta) = (x\Delta^m) = x_m \qquad (m = 0, 1, \ldots),$$
$$(\Delta\cdots\Delta F\Delta\cdots\Delta) = (\Delta^n F\Delta^m) = F_m^{(n)} \quad (m, n = 0, 1, \ldots),$$
$$(\Delta\cdots\Delta P\Delta\cdots\Delta) = (\Delta^n P\Delta^m) = P_m^{(n)} \quad (m, n = 0, 1, \ldots)$$

are called, respectively, *object, operator* and *predicate variables.**) The subscript m labels the *number* of a variable while n is the *arity* of the corresponding operator or predicate variable. The symbols x_m, $F_m^{(n)}$ and $P_m^{(n)}$ are to be viewed as abridged notation for words in the second-order language. The equality sign will play a dual role in the sequel. On the one hand, it will be an alphabet symbol of the second-order language and on the other, it will be the equality sign used in the metalanguage in which the properties of words in the second-order language are being expressed.

The above second-order language alphabet is finite. However, each of the collections: object variables, operator variables of arbitrary arity or predicate variables of arbitrary arity is infinite. Nevertheless, at times the alphabet of the second-order language will be augmented further by symbols playing the role of object variables, operator variables or predicate variables.

Not all of the words in the second-order language will be used to express properties of algebraic systems. The ones that have this function are called *terms* and *formulas*. In this subsection, we shall give

---

*) The first two are also called *individual* and *function* variables [Trans.].

the definition and simplest properties of terms and in the next subsection, those of formulas.

By definition,

(a) each word of the form x_i or $F_i^{(0)}$ is a term;

(b) if $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are terms, then the word $F_i^{(n)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is also a term;

(c) a word is a term if it is a term by virtue of conditions (a) and (b).

Thus, for example, the words

$$F_1^{(3)}\big(F_1^{(0)}, x_2, F_2^{(1)}(x_1)\big), \quad F_1^{(2)}\big(F_2^{(2)}(x_1, x_2), F_1^{(2)}(x_1, x_1)\big)$$

are terms, but the words

$$F_1^{(3)} F_1^{(0)}\big(x_2, F_1^{(1)}(x_1)\big), \quad F_1^{(2)}(P_1^{(0)}, F_1^{(0)})$$

are not.

We now wish to define what is meant by *value of a term* having assigned values to the operator and object symbols occurring in the term. To assign the value of an n-ary variable $F_i^{(n)}$ means to associate with it a specific n-ary operation defined on a fixed "carrier" A. To assign a value to the variable x_1 means to associate with it some element a_1 of the carrier A. Such a set A is stipulated in advance and the values of all operator and object symbols are defined in A. The value of any term T is then defined by induction on the number of operator symbols occurring in T. If a term is of the form x_i or $F_i^{(0)}$, its value coincides with that of the variable x_i or $F_i^{(0)}$. But if T is of the form $F_i^{(n)}(T_1, \ldots, T_n)$ and the terms $T_1, \ldots, T_n$ have the respective values $a_1, \ldots, a_n$ in A, then the value of T is the value of $F_i^{(n)}$ when applied to $a_1, \ldots, a_n$. In other words, *a term is evaluated by applying to the values of the object variables all those operations specified in the term.* For example, suppose we wish to evaluate the term

$$F_2^{(2)}\big(x_1, F_1^{(2)}(x_1, x_2)\big), \tag{2}$$

when x_1 and x_2 are respectively 2 and 3 and $F_1^{(2)}$ and $F_2^{(2)}$ "denote" the operations $+$ and $\cdot$ on the set $N = \{0, 1, 2, \ldots\}$. We find in succession that

$$F_1^{(2)}(x_1, x_2) = 2 + 3 = 5,$$

and

$$F_2^{(2)}\big(x_1, F_1^{(2)}(x_1, x_2)\big) = 2 \cdot 5 = 10.$$

Similarly, if $F_1^{(2)}$ stands for addition of natural numbers and $F_2^{(2)}$ for exponentiation $(x_1^{x_2})$, then for the same values of the object variables, term (2) has the value $2^{2+3} = 32$.

At times, the value of an operator symbol will be allowed to be a partial operation (and hence not defined for all values of the object

variables). The value of a term may then be indeterminate. In other words, the value of a term involving partial operator symbols is well-defined if and only if the successive application of each operation occurring in the term always yields a definite value when applied to the values of the object variables. As an example, consider the partial algebra $\langle\{0, 1, 2, \ldots\}; +, \cdot, -\rangle$, in which $+$ and $\cdot$ are the ordinary operations of addition and multiplication and $y - x$ is the partial operation of subtraction only defined, of course, for pairs x and y such that $x \leqq y$. We then have

$$1 + 0 \cdot (2 - 1) = 1, \quad (2 - 3) + 3 = \text{indeterm.},$$
$$1 + 0 \cdot (1 - 2) = \text{indeterm.}$$

The following theorem states one of the most striking properties of terms.

Theorem 1. *Let α be a homomorphism of an algebraic system $\mathfrak{A} = \langle A, \Omega\rangle$ into an equitype system $\mathfrak{B} = \langle B, \Omega'\rangle$ and let $T(x_1, \ldots, x_r)$ be a term involving operator variables $F_1^{(m_1)}, \ldots, F_s^{(m_s)}$ and object variables $x_1, \ldots, x_r$. Let the values of $F_1^{(m_1)}, \ldots, F_s^{(m_s)}$ in $\mathfrak{A}$ be operations $F_1, \ldots, F_s$ belonging to Ω and let the values of these variables in $\mathfrak{B}$ be the like operations $G_1, \ldots, G_s$ belonging to Ω'. Then for any elements $a_1, \ldots, a_r$ in $\mathfrak{A}$,*

$$T(a_1, \ldots, a_r)\alpha = T(a_1\alpha, \ldots, a_r\alpha), \tag{3}$$

where the left-hand and right-hand members are the respective values of T in $\mathfrak{A}$ and $\mathfrak{B}$ for the specified values of the variables.

We shall prove this by induction on the number of embedments of operator symbols in T. Let n denote the number of embedments. If n is 0 or 1, then T has one of the forms

$$x_i, F_i^{(0)}, F_i^{(m)}(\mathfrak{x}_i, \ldots, \mathfrak{x}_m),$$

where $\mathfrak{x}_1, \ldots, \mathfrak{x}_m \in \{x_1, \ldots, x_r\}$. In each of these instances, equation (3) coincides with equation (1) of Section 2.2 in the definition of homomorphism.

Now let term T have n embedments of operator symbols, $n > 1$, and suppose (3) holds for terms with a fewer number of embedments. T is of the form $F_i^{(m_i)}(T_1, \ldots, T_{m_i})$, in which the terms T_{m_j} have fewer than n embedments. By definition of homomorphism, we have

$$F_i(T_1, \ldots, T_{m_i})\alpha = G_i(T_1\alpha, \ldots, T_{m_i}\alpha).$$

By the induction assumption,

$$T_j(a_1, \ldots, a_r)\alpha = T_j(a_1\alpha, \ldots, a_r\alpha)$$

and so

$$\begin{aligned} &F_i(T_1, \ldots, T_{m_i})\alpha \\ &= G_i\big(T_1(a_1\alpha, \ldots, a_r\alpha), \ldots, T_{m_i}(a_1\alpha, \ldots, a_r\alpha)\big) \\ &= T(a_1\alpha, \ldots, a_r\alpha), \end{aligned}$$

as required.

In Section 2.3, we introduced the notion of generating set of a subsystem A_1 of an algebraic system $\mathfrak{A}$. The notion may be made more concrete with the help of the notion of term. Let us agree to say that a term T *has the signature of system* $\mathfrak{A}$ (see also Section 6.3) if each operator variable in T has been assigned a value in $\mathfrak{A}$ corresponding to a basic operation in $\mathfrak{A}$. We shall say that term T *is in the elements* $a_1, \ldots, a_r$ *of* $\mathfrak{A}$ if each object variable in T has been assigned a value corresponding to one of the $a_1, \ldots, a_r$. Thus, the value of each term with the signature of $\mathfrak{A}$ and in elements of $\mathfrak{A}$ is some element of $\mathfrak{A}$.

Theorem 2. *A subset S of an algebraic system $\mathfrak{A} = \langle A, \Omega\rangle$ is a generating set for a subsystem $\mathfrak{A}_1$ if and only if each element of $\mathfrak{A}_1$ is the value of some term with the signature of $\mathfrak{A}$ and in elements of S.*

First of all, $\mathfrak{A}_1$ contains its generating set S and is closed under basic operations. Thus, when evaluating a subterm in elements of S, we remain inside $\mathfrak{A}_1$ and so the final value of a term is also in $\mathfrak{A}_1$. Conversely, suppose $\mathfrak{M}$ is the set of values of all possible terms in elements of S. $\mathfrak{M}$ is closed under each basic operation F_ξ. To see this, let $u_1, \ldots, u_{m_\xi}$ be the values of terms $T_1, \ldots, T_{m_\xi}$ in elements of S. Then $F_\xi(u_1, \ldots, u_{m_\xi})$ will be the value of the term $F_\xi(T_1, \ldots, T_{m_\xi})$ in the same elements of S and so $F_\xi(u_1, \ldots, u_{m_\xi}) \in \mathfrak{M}$. By definition, $\mathfrak{A}_1$ is the smallest set containing S and it is closed under basic operations. Therefore $\mathfrak{A}_1 \subseteq \mathfrak{M}$. But above we proved that $\mathfrak{M} \subseteq \mathfrak{A}_1$. Hence, $\mathfrak{A}_1 = \mathfrak{M}$, as required.

Corollary 1. *If α and β are homomorphisms of an algebraic system $\mathfrak{A}$ into $\mathfrak{B}$ coinciding on a generating set S for $\mathfrak{A}$, then $\alpha = \beta$.*

By Theorem 2, each element $x \in \mathfrak{A}$ is the value of an appropriate term $T(a_1, \ldots, a_r)$, where $a_1, \ldots, a_r$ are elements of S. By hypothesis,

$$a_i\alpha = a_i\beta \qquad (i = 1, \ldots, r).$$

Theorem 1 thus implies

$$T(a_1, \ldots, a_r)\alpha = T(a_1, \ldots, a_r)\beta,$$

i.e., $x\alpha = x\beta$ for each $x \in \mathfrak{A}$.

Hence, in order to define a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, we merely have to specify what the elements $a_i\alpha$ are ($a_i \in S$). A question however is: What conditions must be satisfied by arbitrarily prescribed elements $a\alpha$ so that a mapping of S into $\mathfrak{B}$ is extendable to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$?

Theorem 3. *Let S be a generating set for algebra $\mathfrak{A}$ and let α^* be a mapping of S into an equitype algebra $\mathfrak{B}$. Then α^* is extendable to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ if and only if*

$$\begin{aligned} T_1(a_1, \ldots, a_r) = T_2(a_1, \ldots, a_r) \\ \Rightarrow T_1(a_1\alpha^*, \ldots, a_r\alpha^*) = T_2(a_1\alpha^*, \ldots, a_r\alpha^*) \end{aligned} \tag{4}$$

for any terms T_1 and T_2 having the signature of $\mathfrak{A}$ and any $a_1, \ldots, a_r$ in S.

It follows from Theorem 1 that (4) is a necessary condition. Conversely, suppose (4) holds. Choose any element $a \in \mathfrak{A}$. By Theorem 2, there is a term $T(x_1, \ldots, x_r)$ such that $a = T(a_1, \ldots, a_r)$ for suitable $a_1, \ldots, a_r$ in S. Now define a mapping α of $\mathfrak{A}$ by

$$a\alpha = T(a_1\alpha^*, \ldots, a_r\alpha^*). \tag{5}$$

Condition (4) assures that $a\alpha$ is independent of the choice of T, i.e., α is a single-valued mapping of $\mathfrak{A}$ into $\mathfrak{B}$. Clearly, $\alpha = \alpha^*$ on S. Choosing any $z_1, \ldots, z_{m_\xi}$ in $\mathfrak{A}$ and expressing them as terms in elements of S, we obtain from (5) the relation

$$F_\xi(z_1, \ldots, z_{m_\xi})\alpha = G_\xi(z_1\alpha, \ldots, z_{m_\xi}\alpha),$$

which shows that α is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Theorem 3 concerns homomorphisms of algebras. For homomorphisms of systems, it is necessary to tack on condition (2) of Section 2.2. In the present situation, it takes the form

$$P_\eta(z_1, \ldots, z_{m_\eta}) \Rightarrow Q_\eta(z_1\alpha^*, \ldots, z_{m_\eta}\alpha^*)$$

for any terms $z_1, \ldots, z_{m_\eta}$ in elements of S.

6.2 Formulas. Several notions that will play the same kind of fundamental role in the second-order language as term and value of a term are formula, free variable, bound variable and value of a formula. We define *formula*, *free variable* and *bound variable* inductively as follows:

(a) To each predicate variable $P_i^{(n)}$ and set of terms $T_1, \ldots, T_n$, the word $P_i^{(n)}(T_1, \ldots, T_n)$ is a formula. Each 0-ary predicate word $P_i^{(0)}$ is a formula. All variables in such formulas are free.

(b) For each pair of terms T_1 and T_2, the word $T_1 = T_2$ is a formula. All variables in such a formula are free.

(c) If $\mathfrak{F}$ is a formula, then $\neg\mathfrak{F}$ is also a formula. All free variables in $\mathfrak{F}$ are also free in $\neg\mathfrak{F}$. All bound variables in $\mathfrak{F}$ are also bound in $\neg\mathfrak{F}$.

(d) Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be formulas such that variables occurring simultaneously in both formulas are free in each of them. Then the words

$$(\mathfrak{F}_1 \,\&\, \mathfrak{F}_2), \quad (\mathfrak{F}_1 \vee \mathfrak{F}_2), \quad (\mathfrak{F}_1 \to \mathfrak{F}_2) \tag{1}$$

are also formulas. Variables that are free in at least one of the formulas $\mathfrak{F}_1$ or $\mathfrak{F}_2$ are also free in each formula in (1). Variables that are bound in either $\mathfrak{F}_1$ or $\mathfrak{F}_2$ are also bound in each formula in (1).

(e) Let $\mathfrak{x}$ denote any of the variables x_i, $F_j^{(i)}$ or $P_j^{(i)}$ and let $\mathfrak{x}$ occur freely in a formula $\mathfrak{F}$. Then the words

$$(\forall \mathfrak{x})\mathfrak{F}, \quad (\exists \mathfrak{x})\mathfrak{F} \tag{2}$$

are again formulas in which $\mathfrak{x}$ is a bound variable and all other free or bound variables in $\mathfrak{F}$ are of the same nature in each formula in (2).

(f) A word is called a formula if it is a formula in consequence of assertions (a)—(e).

The inductive definition (a)—(f) is with respect to the number of symbols $P_j^{(i)}$ and $=$ occurring in the formulas.

There is a very simple decision procedure for determining whether a word is a term or formula. To this end, observe first that parentheses always occur in a term or formula as paired corresponding left and right parentheses and that two pairs of parentheses cannot separate each other. A pair of parentheses is *outermost* if all other letters of a word lie inside this pair.

From the rules of formation of terms and formulas, it follows that the only terms and formulas not containing parentheses are the words

$$\begin{gathered} x_j,\ F_i^{(0)},\ P_i^{(0)},\ x_i = x_j,\ x_i = F_j^{(0)},\ F_i^{(0)} = F_j^{(0)}, \\ \neg x_i = x_j,\quad \neg\neg x_i = x_j,\quad \neg x_i = F_j^{(0)}, \end{gathered} \tag{3}$$

etc.

Let $\mathfrak{A}$ be a word containing parentheses. We can find innermost pairs of parentheses in $\mathfrak{A}$. If $\mathfrak{A}$ is to be a formula, there must be a word of form (3) inside an innermost pair or else $\mathfrak{A}$ must contain along with such an innermost pair a subword of the form $F_j^{(i)}(u_1, \ldots, u_i)$ or $P_j^{(i)}(u_1, \ldots, u_i)$, where $u_1, \ldots, u_i$ are variables of the form x_a or $F_a^{(0)}$. A pair of parentheses in a word is said to be of higher order than a second pair if the second pair is contained in the first. Innermost pairs of parentheses are the pairs of lowest order. The passage from a pair of

parentheses inside word $\mathfrak{A}$ to pairs of higher order must be executed in accord with rules (a)—(e). If the rules are violated at any time, word $\mathfrak{A}$ is not a formula. For example, the words

$$(\forall x_1)\left(x_1 = x_2 \,\&\, (\exists x_3) P_1^{(2)}(x_2, x_3)\right), \quad (\exists x_1)(\forall x_2) \neg x_1 = x_2$$

are formulas, but the words

$$(\forall x_1) F_2^{(1)}(x_1, x_2), \quad \left((\forall x_1) P_1^{(1)}(x_1) \,\&\, (\exists x_1) P_1^{(1)}(x_1)\right)$$

are not.

The *value* of a predicate variable $P_j^{(i)}$ on a set A is any specific i-ary predicate P defined on A. The *value of a formula* $\mathfrak{A}$ when all its free object, operator and predicate variables have been assigned values in A is T or F as determined by the following rules:

(a′) Let $\mathfrak{A}$ be a formula of the form $u = v$ with u and v terms. The values of u and v are elements a and b in A. If $a = b$, the value of $\mathfrak{A}$ is T. But if $a \neq b$, its value is F.

(b′) Let $\mathfrak{A}$ be a formula of the form $P_j^{(i)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_i)$, where $\mathfrak{a}_1, \ldots, \mathfrak{a}_i$ are terms. The values of the terms are any elements $a_1, \ldots, a_i$ of A. The value of the variable $P_j^{(i)}$ is some i-ary predicate P defined on A. By definition, the value of $P_j^{(i)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_i)$ is whatever value $P(a_1, \ldots, a_i)$ has.

(c′) Let $\mathfrak{F}$ be a formula of the form $\neg \mathfrak{A}$, where $\mathfrak{A}$ is a shorter formula. Evaluate $\mathfrak{A}$. If it is equal to T, the value of $\mathfrak{F}$ is F. If the value of $\mathfrak{A}$ is F, then the value of $\mathfrak{F}$ is T.

(d′) If the formula $\mathfrak{F}$ is one of the following:

$$(\mathfrak{A} \,\&\, \mathfrak{B}), \quad (\mathfrak{A} \vee \mathfrak{B}), \quad (\mathfrak{A} \to \mathfrak{B})$$

and the values of $\mathfrak{A}$ and $\mathfrak{B}$ are given, then $\mathfrak{F}$ is evaluated according to the familiar truth table

$\mathfrak{A}$	$\mathfrak{B}$	$(\mathfrak{A} \,\&\, \mathfrak{B})$	$(\mathfrak{A} \vee \mathfrak{B})$	$(\mathfrak{A} \to \mathfrak{B})$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

(4)

(e′) Let $\mathfrak{F}$ be a formula of the form $(\forall x_i)\mathfrak{A}$ or $(\exists x_i)\mathfrak{A}$. By hypothesis, the values of all free variables in $\mathfrak{F}$ are assigned. The free variables in $\mathfrak{A}$ are all free variables in $\mathfrak{F}$ plus the variable x_i whose value is unassigned. Since $\mathfrak{A}$ is a shorter formula than $\mathfrak{F}$, by assigning some value a from set A to x_i we can evaluate $\mathfrak{A}$. If the value of $\mathfrak{A}$ is T for each a in A, then the

value of $(\forall x_i)\mathfrak{A}$ is T. Otherwise the value of $(\forall x_i)\mathfrak{A}$ is taken to be F. If there is value of x_i in A for which $\mathfrak{A}$ is T, then the value of $(\exists x_i)A$ is taken to be T. If the value of $\mathfrak{A}$ is F for all values of x_i in A. then the value of $(\exists x_i)\mathfrak{A}$ is taken to be F.

(f') If $\mathfrak{F}$ is a formula of the form $(\forall F_j^{(i)})\mathfrak{A}$, $(\exists F_j^{(i)})\mathfrak{A}$, $(\forall P_j^{(i)})\mathfrak{A}$ or $(\exists P_j^{(i)})\mathfrak{A}$, then the value of $\mathfrak{F}$ is determined just as in case (e'). The only difference is that the values of the variables $F_j^{(i)}$ and $P_j^{(i)}$ are now taken to be operations and predicates defined on A.

$\mathfrak{F}$ is said to be a *true formula* in a given set A if for assigned values of all its free variables the value of $\mathfrak{F}$ is T. A formula $\mathfrak{F}$ is *false* if its value is F.

Consider, for example, the formula

$$(\exists x_1)F_1^{(2)}(x_1, x_2) = x_3. \tag{5}$$

To find its value we must first specify the basic set A and then assign values to the object variables and operator variable $F_1^{(2)}$. Suppose A is the set of natural numbers and $F_1^{(2)} = +$. Then the truth of formula (5) is equivalent to the truth of the relation

$$(\exists x_1)x_1 + x_2 = x_3.$$

Thus, formula (5) is true, for example, when $x_2 = 2$ and $x_3 = 5$ but false when $x_2 = 5$ and $x_3 = 2$.

Consider the formula

$$(\forall x_1)\,(\exists x_2)F_1^{(2)}\big(x_2, F_2^{(2)}(x_1, x_3)\big) = x_3. \tag{6}$$

on the set $A = \{0, 1, 2, \ldots\}$. Let the value of $F_1^{(2)}$ be the operation of addition and that of $F_2^{(2)}$ the operation of multiplication. Then the truth of formula (6) is equivalent to the truth of the assertion that the equation

$$y + xa = a \tag{7}$$

has at least one solution y in A corresponding to each x in A. This is obviously true if $a = 0$ and false if $a \neq 0$.

With the same values for the variables, the truth of the formula

$$(\exists x_2)\,(\forall x_1)F_1^{(2)}\big(x_2, F_2^{(2)}(x_1, x_3)\big) = x_3$$

is equivalent to the assertion that there is a y in A for which equation (7) holds for all values of x in A.

Various kinds of abbrevations are usually employed to write down formulas in the second-order language. The most useful of these are the

following. First of all, outermost parentheses are omitted. For example, the word

$$x_1 = x_2 \,\&\, \neg\, x_1 = x_3$$

stands for the formula

$$(x_1 = x_2 \,\&\, \neg\, x_1 = x_3).$$

Instead of the formula $\neg\, x = y$, one writes $x \neq y$. When there are strings of the quantifier $\forall x$, the intermediary symbols are omitted and instead of $(\forall x_i)\ (\forall x_j)\ \ldots\ (\forall x_k)\,\mathfrak{A}$, one writes $(\forall x_i x_j \ldots x_k)\,\mathfrak{A}$. Similarly, instead of $(\exists x_i)\ (\exists x_j)\ \ldots\ (\exists x_k)\,\mathfrak{A}$, one writes $(\exists x_i x_j \ldots x_k)\,\mathfrak{A}$. In formulas of the form,

$$(\ldots((\mathfrak{A}_1 \,\&\, \mathfrak{A}_2) \,\&\, \mathfrak{A}_3) \,\&\, \ldots \,\&\, \mathfrak{A}_k),$$
$$(\ldots((\mathfrak{A}_1 \vee \mathfrak{A}_2) \vee \mathfrak{A}_3) \vee \ldots \vee \mathfrak{A}_k)$$

the parentheses are omitted and one writes

$$\mathfrak{A}_1 \,\&\, \mathfrak{A}_2 \,\&\, \mathfrak{A}_3 \,\&\, \ldots \,\&\, \mathfrak{A}_k, \qquad \mathfrak{A}_1 \vee \mathfrak{A}_2 \vee \mathfrak{A}_3 \vee \ldots \vee \mathfrak{A}_k.$$

A still further reduction in the number of parentheses can be achieved by using the *priority rule for operations* which says that the connective $\vee$ has priority over the connective $\&$ and the connective $\&$ has priority over the connective $\rightarrow$. This means that when parentheses are absent they are to be restored in the following order: first pick out the smallest segment which contains the first symbol $\vee$ and such that enclosing it in parentheses converts the segment into a formula. Having restored all pairs of parentheses in this way corresponding to the symbol $\vee$, proceed to restore the parentheses associated with $\&$ and then those associated with $\rightarrow$. For example, applying this rule, we see that the word

$$\neg\, x_1 = x_2 \,\&\, \neg\, P_1^{(0)} \vee P_2^{(0)} \rightarrow x_1 = x_3 \tag{8}$$

is an abridgement of the formula

$$\big((\neg\, x_1 = x_2 \,\&\, (\neg\, P_1^{(0)} \vee P_2^{(0)})\big) \rightarrow x_1 = x_3\big),$$

These abridgements make it possible to avoid unnecessary accumulations of parentheses. Apart from this technique, there is the important way of condensing notation by the introduction of new notation. Suppose $\mathfrak{A}$ contains a subformula $\mathfrak{B}$ having object variables x_i and x_j that are free in $\mathfrak{A}$. We can introduce the notation $U(x_i, x_j)$ for $\mathfrak{B}$, where U is a symbol not occurring in the alphabet of our object language. If we replace $\mathfrak{B}$ in formula $\mathfrak{A}$ by the word $U(x_i, y_j)$, we obtain a new word

said to be the abridged notation for $\mathfrak{A}$. Such abridgements of notation may be introduced not only for subformulas but also for individual variables. For instance, we shall say: choose a binary operation L, unary predicates Q and R, and object variables x and y and consider the formula

$$L(x, y) = L(y, x) \,\&\, \neg Q(x) \,\&\, R(y). \tag{9}$$

This means that we are considering a formula obtainable from word (9) by substituting any *different* object variables x_i and y_j for x and y, an operator variable $F_1^{(2)}$ for L and different unary predicate variables $P_s^{(1)}$ and $P_t^{(1)}$ for Q and R.

As already indicated in Section 3.1, terms such as $F_j^{(i)}(x_{m_1}, \ldots, x_{m_i})$ will be denoted at times by $x_{m_1} \ldots x_{m_i} F_j^{(i)}$ and formulas such as $P_j^{(i)}(x_{m_1}, \ldots, x_{m_i})$ by $x_{m_1} \ldots x_{m_i} P_j^{(i)}$. If the operations and predicates are binary, xFy and xPy will often be used instead of $F(x, y)$ and $P(x, y)$. As a result, rather long formulas such as, for example,

$$P\big(F(x, y), z\big) \,\&\, F(x, z) = F(z, x),$$

are capable of being written in the shorter form

$$(xFy)Pz \,\&\, xFz = zFx.$$

In speaking of the value of a formula or term, we first have to specify the underlying set A and we then have to assign values from A to the free variables. It is sometimes of interest to fix the values of certain free variables defined on A and then to observe how the values of the formula or word vary as the values of the remaining free variables are allowed to vary. When that is the case, we shall introduce notation for the fixed free variable values and we shall consider the mixed type formula consisting partially of second-order language symbols and partially of notation for the specific elements, operations and predicates serving as values for corresponding variables in the formula.

For example, instead of saying: consider the values of the formula

$$(\forall x_2)\big(F_1^{(2)}(x_1, x_2) = x_2 \,\&\, F_2^{(2)}(x_1, x_2) = x_1\big)$$

in the set $N = \{0, 1, 2, \ldots\}$, where $F_1^{(2)}$ is the operation of addition and $F_2^{(2)}$ the operation of multiplication and x_i assumes values from N, we shall say: consider the values of the formula

$$(\forall y)\,(x + y = y \,\&\, xy = x) \tag{10}$$

in $\{0, 1, 2, \ldots\}$ where $+$ and $\cdot$ are the ordinary operations of addition and multiplication. The rules for evaluating a formula immediately imply

that (10) is a true formula if and only if $x = 0$. In similar fashion, it is easy to see that the formula

$$(\exists z)\,(x + z = y \,\&\, x \neq y) \tag{11}$$

is a true formula if and only if $x < y$. It is likewise easy to see that

$$(\forall yz)\,(yy \neq y \,\&\, zz \neq z \to x \neq yz) \,\&\, xx \neq x \tag{12}$$

is a true formula if and only if x is a prime number.

Let formula (12) be denoted by $B(x)$ and consider the formula

$$(\forall x)\,\big(xx \neq x \to (\exists yz)\,(B(y) \,\&\, B(z) \,\&\, x + x = y + z)\big). \tag{13}$$

This is a true formula if and only if each even number greater than two is the sum of two primes. The truth of this assertion is an open question at the moment. Thus, today the value of formula (13) is unknown.

If instead of N, the carrier is taken to be set of all real numbers, then the values of formulas (10)—(13) will be different though as before (10) on the set of reals also "determines" the number 0. That is to say, formula (11) will be true for all $x \neq y$ and formulas (12) and (13) false for all x. To define the relation $x < y$ using addition and multiplication, we may replace (11) by the formula

$$(\exists z)\,\big(xx + (zz) = y \,\&\, z + z \neq z\big),$$

since the square of a nonzero real number is positive.

6.3 Second-Order Properties. Many important properties of algebraic systems have a perfectly natural description in the second-order language. Many problems also can be formulated in the second-order language in a perfectly natural way. A simple example is the above Goldbach-Euler conjecture that every even number $\geqq 4$ is representable as the sum of two primes. The general idea underlying the application of the second-order language to the theory of algebraic systems is as follows.

Consider an arbitrary class $\mathfrak{K}$ of algebraic systems of given type $\langle m_0, \ldots, m_\xi, \ldots; n_0, \ldots, n_\eta, \ldots\rangle$ $(\xi < \alpha,\ \eta < \beta)$. With each such type is associated a set σ of operator and predicate symbols $F_0, \ldots, F_\xi, \ldots$, $P_0, \ldots, P_\eta, \ldots$ of arity $m_0, \ldots, m_\xi, \ldots, n_0, \ldots, n_\eta, \ldots$, respectively. The symbols $F_0, \ldots, F_\xi, \ldots, P_0, \ldots, P_\eta, \ldots$ are to be viewed as generic names for the corresponding operations and predicates in the systems belonging to $\mathfrak{K}$. We shall call the collection of such symbols the *signature* of $\mathfrak{K}$. Let $\mathfrak{A}$ be a given arbitrary system in $\mathfrak{K}$. The values that the symbols

F_ξ and P_η have in $\mathfrak{A}$ are said to be the ξ-th operation and η-th predicate of this system. Thus, defining a system $\mathfrak{A}$ of signature σ on A is the same as specifying a nonempty set A and associating with each signature symbol a value corresponding to an operation or predicate on A of appropriate arity. We shall say that a formula in the second-order language *has signature* σ if all its free operator and predicate symbols belong to σ. A formula containing no free object variables is said to be *closed.**) An *absolutely closed* formula is one containing no free object, operator and predicate variables.

For each absolutely closed formula $\mathfrak{F}$ and each set A, we may ask the question: What is the truth value of $\mathfrak{F}$ in A? To answer it, we need know no additional information about the values of free variables since there are none in $\mathfrak{F}$. For example, the formulas

$$(\forall P)\,(\forall x)\,\big(P(x) \vee \neg P(x)\big), \qquad (\forall F)\,(\forall x)\,\big(F(x) = F(x)\big),$$

in which P and F are unary predicate and operator symbols, are true in any nonempty set.

Conversely, to question the truth or falsity of a nonabsolutely closed formula $\mathfrak{F}$, we must first assign values to its free variables. However, all free variables in a closed formula of signature σ have fixed values in each algebraic system of signature σ. Consequently, we can again speak of the truth or falsity of $\mathfrak{F}$ except not in the set but in the system of signature σ. For instance, the formula

$$(\forall xyz)\,\big(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\big) \tag{1}$$

is a closed formula of signature $\sigma = \{+, \cdot\}$, where $+$ and $\cdot$ symbolize binary operator symbols. The set $N = \{0, 1, 2, \ldots\}$ with the ordinary operations of addition and multiplication is an algebra of signature σ. The formula (1) is clearly true in this algebra. On the other hand, the formula

$$(\forall x)\,(x \cdot x = x) \tag{2}$$

is false in it.

Let $S(x) = x + 1$ and consider the algebra $\langle N, S\rangle$. The formula

$$(\forall P)\,(\forall x)\,\Big((\forall y)\,\big(x \neq S(y)\big) \,\&\, P(x) \,\&\, (\forall z)\,\big(P(z) \to P\big(S(z)\big)\big) \to (\forall u)\,P(u)\Big) \tag{3}$$

expresses the principle of mathematical induction and so is true in the algebra $\langle N, S\rangle$.

---

*) A closed formula is sometimes called a *sentence*. To every well formed formula containing free variables there corresponds a formula containing no free object variables called the *closure* [*Trans.*].

Let us consider one further example. The formula

$$(\forall z)\,(x + y \cdot z = x) \tag{4}$$

is a formula of signature $\{+, \cdot\}$ but it is not closed. Its truth value in the algebra $\langle N; +, \cdot\rangle$ depends on what values the object variables x and y have in N. If $x = 1$ and $y = 0$, formula (4) is true. If $x = y = 1$, formula (4) is false, etc. We see that formula (4) associates with each pair of numbers x and y one of the elements of the set $\{\mathsf{T}, \mathsf{F}\}$ and thus it defines a binary predicate on N.

The values of (4) may be studied not only in algebras of signature $\{+, \cdot\}$ but also in algebras having a more restricted signature, for example, in $\langle N, \cdot\rangle$. In that event, the values of (4) will depend not only on the values of the object variables x and y but also on the choice of operation $+$. In other words, the value of formula (4) will be a predicate defined on the cartesian product $N \times N \times T$, where T is the family of all binary operations defined on N. Such predicates are called *second-order predicates*. In contrast to them, the ordinary predicates, which we shall only be considering, are sometimes called *first-order predicates*.

As already stipulated, the truth or falsity of a closed formula of signature σ in a system of signature σ does not depend on any additional data and so is a property of the system itself. The fact that a given closed formula $\mathfrak{F}$ of signature σ is true in a system $\mathfrak{A}$ of signature σ is said to be a *second-order property*.

We wish to show that second-order properties are abstract properties, i.e., if a closed formula $\mathfrak{F}$ of signature σ is true or false in some system $\mathfrak{A}$ of signature σ, then $\mathfrak{F}$ will have the same value in any system $\mathfrak{B}$ isomorphic to $\mathfrak{A}$. In fact, we can prove the following more general theorem.

Theorem 1. *Let α be an isomorphic mapping of an algebraic system $\mathfrak{A}$ of signature σ onto a system $\mathfrak{B}$. Let $\mathfrak{F}(u_1, \ldots, u_n)$ be any formula of signature σ having free object variables $u_1, \ldots, u_n$. Then for arbitrary $u_1, \ldots, u_n$ in $\mathfrak{A}$*

$$\mathfrak{F}(u_1, \ldots, u_n) \Leftrightarrow \mathfrak{F}(u_1\alpha, \ldots, u_n\alpha). \tag{5}$$

The theorem is most readily proved by induction on the number r of embedments in $\mathfrak{F}$ of the logical symbols $\&$, $\vee$, $\neg$, $\rightarrow$, $\exists$, $\forall$, $=$ and predicate symbols. If $r = 1$, $\mathfrak{F}$ has one of the forms

$$\mathfrak{a}_1 = \mathfrak{a}_1, \quad \mathfrak{a}_1 = \mathfrak{a}_2, \quad P^{(0)}, \quad P^{(0)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n), \tag{6}$$

where $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are terms in $u_1, \ldots, u_n$. In this case, relation (5) is a consequence of the definition of isomorphism (see Section 2.2) and Theorem 1 of Section 6.1.

Suppose now that $r > 1$ for formula $\mathfrak{F}$ and that Theorem 1 is true for formulas having a fewer number of embedments than r. In conformity with the definition of formula, consider all the ways in which $\mathfrak{F}$ could have arisen from formulas with a fewer number of embedments than r.

(a) $\mathfrak{F}$ has the form $\neg\mathfrak{F}_1$. By hypothesis, $\mathfrak{F}_1(u) \Leftrightarrow \mathfrak{F}_1(u\alpha)$ and so $\neg\mathfrak{F}_1(u) \Leftrightarrow \neg\mathfrak{F}_1(u\alpha)$.

(b) $\mathfrak{F}$ has one of the forms

$$\mathfrak{F}_1 \& \mathfrak{F}_2, \quad \mathfrak{F}_1 \vee \mathfrak{F}_2, \quad \mathfrak{F}_1 \to \mathfrak{F}_2,$$

wherein $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are formulas corresponding to a smaller value of r than for $\mathfrak{F}$. By hypothesis,

$$\mathfrak{F}_i(u) \Leftrightarrow \mathfrak{F}_i(u\alpha) \qquad (i = 1, 2)$$

and hence, relation (5) is valid.

(c) $\mathfrak{F}$ has the form $(\forall x)\mathfrak{F}_1(u_1, \ldots, u_n, x)$. By hypothesis,

$$\mathfrak{F}_1(u_1, \ldots, u_n, x) \Leftrightarrow \mathfrak{F}_1(u_1\alpha, \ldots, u_n\alpha, x\alpha) \tag{7}$$

for each x in $\mathfrak{A}$. If $(\forall x)\mathfrak{F}_1(u_1, \ldots, u_n, x)$ is a true formula, then the left-hand formula in (7) and hence the right-hand one is true for each x in $\mathfrak{A}$. But to each element y in $\mathfrak{B}$, there is an x in $\mathfrak{A}$ such that $x\alpha = y$. Therefore $\mathfrak{F}_1(u_1\alpha, \ldots, u_n\alpha, y)$ is a true formula for each y in $\mathfrak{B}$. In other words, $(\forall x)\mathfrak{F}_1(u_1\alpha, \ldots, u_n\alpha, x)$ is a true formula and so relation (5) is proved. But if $(\forall x)\mathfrak{F}_1(u_1, \ldots, u_n, x)$ is false, then $\mathfrak{F}_1(u_1, \ldots, u_n, x)$ is false for some x in $\mathfrak{A}$. From (7) we conclude that $\mathfrak{F}_1(u_1\alpha, \ldots, u_n\alpha, y)$ is false for $y = x\alpha$. Thus, $(\forall x)\mathfrak{F}_1(u_1\alpha, \ldots, u_n\alpha, x)$ is false in $\mathfrak{B}$. Hence, relation (5) also holds in this case.

(d) $\mathfrak{F}$ has one of the forms

$$(\exists x)\mathfrak{F}_1, \quad (\forall F)\mathfrak{F}_1, \quad (\exists F)\mathfrak{F}_1, \quad (\forall P)\mathfrak{F}_1, \quad (\exists P)\mathfrak{F}_1.$$

In each of these instances, relation (5) can be proved using the same sort of reasoning as in case (c).

The hypotheses of the theorem just proved talk about an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. This cannot be replaced by an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ since the conclusion of the theorem need not hold. Consider, for example, the models $\mathfrak{A} = \langle\{1, 2\}; <\rangle$ and $\mathfrak{B} = \langle\{1, 2, a\}; <\rangle$, where in both cases the predicate $<$ is true just for the pair 1, 2. The identity embedding

$$\alpha\colon\ x \to x \qquad (x = 1, 2)$$

is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Nevertheless, the formula

$$(\forall xy)\ (x \neq y \to x < y \vee y < x),$$

is true in $\mathfrak{A}$ but obviously false in $\mathfrak{B}$.

It is easy to construct an example showing that the conclusion of Theorem 1 is also false for homomorphisms of $\mathfrak{A}$ onto $\mathfrak{B}$. Let $\mathfrak{B} = \langle\{1, 2\}; \leqq\rangle$ and $\mathfrak{A} = \langle\{1, 2, a\}; \leqq\rangle$, where the relation $\leqq$ comprises the pairs $(1, 1)$, $(2, 2)$, (a, a) and $(1, 2)$. The mapping

$$\alpha\colon\ 1 \to 1, \quad 2 \to 2, \quad a \to 2$$

is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. However the formula

$$(\exists xy)\ (\neg x \leqq y\ \&\ \neg y \leqq x)$$

is true in $\mathfrak{A}$ but false in $\mathfrak{B}$.

Theorem 1 for $n = 0$ implies that a necessary condition for an isomorphism to exist between $\mathfrak{A}$ and $\mathfrak{B}$ is that each closed formula of signature σ which is true in one system be true in the other one. Simple considerations with respect to the cardinalities of the systems show that the condition is not sufficient for an isomorphism to exist between $\mathfrak{A}$ and $\mathfrak{B}$. On the other hand, this fact is frequently very useful in proving that two systems $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic. To this end, one merely has to find a closed formula which is true in one system and false in the other.

As an example, consider the models

$$\mathfrak{A} = \langle\{0, 1, 2, \ldots\}; \leqq\rangle, \qquad \mathfrak{B} = \langle\{0, -1, -2, \ldots\}; \leqq\rangle,$$

where $\leqq$ is the customary order relation. Note that the first model has a least element 0 while the second model has none. The first statement can be expressed by the formula

$$(\exists x)\ (\forall y)\ x \leqq y,$$

which thus is true in the first model and false in the second one. Hence, the models are not isomorphic.

Now compare model $\mathfrak{A}$ with the model

$$\mathfrak{C} = \left\langle \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \ldots, \frac{4}{3}, \frac{3}{2}, 2, 3, \ldots \right\}; \leqq \right\rangle.$$

If we were to attempt to separate models $\mathfrak{A}$ and $\mathfrak{C}$ with the help of closed formulas containing no bound predicate or operator variables, we would quickly find out that our attempts would be unsuccessful. However there is a difference in the abstract properties of models $\mathfrak{A}$ and $\mathfrak{C}$. We see that each element x of either model has an immediate successor x'. The relationship between x and x' may be expressed by the formula

$$S(x, x') \overset{df}{=} x \leqq x' \ \&\ x \neq x' \ \&\ (\forall y)\ (x \leqq y\ \&\ y \leqq x' \to x = y \vee y = x').$$

(The notation $\stackrel{df}{=}$ here and throughout the following means that the word on the left-hand side symbolizes the expression on the right-hand side.)

Further, models $\mathfrak{A}$ and $\mathfrak{C}$ both have a least element describable by the formula

$$T(x) \stackrel{df}{=} (\forall y)\, x \leqq y.$$

Finally, we see that as we go from the least element x in $\mathfrak{A}$ to its successor x' and from x' to x'', and so on, we exhaust the entire model. Using the last formula, we can express this property as follows:

$$(\forall P)\,(\forall x)\,\big(T(x) \,\&\, P(x) \,\&\, (\forall y z)\,\big(P(y) \,\&\, S(y, z) \to P(z)\big) \to (\forall u)\, P(u)\big). \tag{8}$$

If we apply the same process to model $\mathfrak{C}$, we clearly exhaust merely the submodel $\left\{0, \frac{1}{2}, \frac{2}{3}, \ldots\right\}$. Thus, formula (8) is true in $\mathfrak{A}$ but false in $\mathfrak{C}$. Therefore $\mathfrak{A}$ and $\mathfrak{C}$ are nonisomorphic models.

There is a profound distinction between the properties of formulas involving bound predicate and operator variables and those of formulas not involving such variables. Formulas in the second-order language not involving bound predicate or operator variables are called *formulas of the applied predicate calculus*, or for brevity, *formulas of the APC* or *APC-formulas*. An APC-formula not involving operator variables is a *formula in the restricted predicate calculus* or an *RPC-formula*. A *formula in the pure predicate calculus* (PPC) is an RPC-formula not involving any equality signs.

The underlying formal language of the theory of algebraic systems is the APC and the underlying language of model theory is the RPC. The languages of the APC, RPC and PPC are basically equivalent to each other. Such languages are said to be of *first-order*. By contrast, the second-order language is much more powerful. Very subtle and logically complex properties are usually not hard to write down in the form of second-order formulas with bound predicate variables. Conversely, even some of the most common ideas such as, for example, subsystem cannot be expressed in the language of the APC. But many definitions and properties are easily expressed in the language of the APC. And what is more important, the fact that a statement can be expressed in the language of the APC will lead to many of the most important properties of that statement. These features of the APC will gradually be presented later on.

6.4 Elementary Theories and Axiomatic Classes. Let $\mathfrak{K}$ be a given class of algebraic systems of signature σ. A closed formula $\mathfrak{F}$ of signature σ

is said to be *true* in $\mathfrak{K}$ if $\mathfrak{F}$ is true in each system belonging to $\mathfrak{K}$. A formula $\mathfrak{F}$ is said to be *satisfiable in* $\mathfrak{K}$ if $\mathfrak{K}$ contains a system in which $\mathfrak{F}$ is true.

The concepts of truth and satisfiability in a class $\mathfrak{K}$ may also be extended to *nonclosed* formulas as follows. Let $\mathfrak{F}$ be a formula involving free predicate variables $P_1, \ldots, P_r$, free operator variables $f_1, \ldots, f_s$ and free object variables $x_1, \ldots, x_t$ that are not in the signature σ. All other free variables in $\mathfrak{F}$ belong to σ. Consider an algebraic system $\mathfrak{M}$ belonging to $\mathfrak{K}$. In order to say anything about the truth or falsity of $\mathfrak{F}$ in $\mathfrak{M}$, we have to assign values to f_i, P_j and x_k $(i, j, k = 1, 2, \ldots)$ from $\mathfrak{M}$ (see Section 6.2). $\mathfrak{F}$ is said to be *true in the class* $\mathfrak{K}$ if it is true in each $\mathfrak{M}$ belonging to $\mathfrak{K}$ for every assignment of values to all its free variables not belonging to the signature of $\mathfrak{K}$. $\mathfrak{F}$ is said to be *satisfiable in* $\mathfrak{K}$ if there is an $\mathfrak{M}$ in $\mathfrak{K}$ in which $\mathfrak{F}$ is a true formula for some assignment of values to all its free variables not in the signature of $\mathfrak{M}$.

An immediate consequence of these definitions is that a formula $\mathfrak{F}$ involving extra-signature free variables $P_1, \ldots, P_r, f_1, \ldots, f_s, x_1, \ldots, x_t$ is true in $\mathfrak{K}$ if and only if the closed formula

$$(\forall P_1 \ldots P_r)\,(\forall f_1 \ldots f_s)\,(\forall x_1 \ldots x_t)\mathfrak{F}$$

is true in $\mathfrak{K}$.

Similarly, a formula $\mathfrak{F}$ having these extra-signature free variables is satisfiable in $\mathfrak{K}$ if and only if the closed formula

$$(\exists P_1 \ldots P_r)\,(\exists f_1 \ldots f_s)\,(\exists x_1 \ldots x_t)\mathfrak{F}$$

is satisfiable in $\mathfrak{K}$.

A second-order formula $\mathfrak{F}$ is said to be *identically true* if it is true in every nonempty set M for every choice of values of its free variables from M. A formula $\mathfrak{F}$ is said to be *satisfiable* if there exists a nonempty set M in which $\mathfrak{F}$ is a true formula for some assignment of values from M to all its free variables.

Identically true formulas of signature σ are clearly true in every class of algebraic systems of signature σ. Conversely, if $\mathfrak{F}$ is a formula of signature σ which is true in the class $\mathfrak{K}$ of *all* algebraic systems of signature σ, then F is identically true.

We mention a few further immediate consequences of the above definitions.

A formula $\mathfrak{F}$ is not satisfiable in class $\mathfrak{K}$ if and only if $\neg\mathfrak{F}$ is a true formula in $\mathfrak{K}$.

The formula $\mathfrak{F}_1 \,\&\, \ldots \,\&\, \mathfrak{F}_m$ is true in class $\mathfrak{K}$ if and only each $\mathfrak{F}_j$ is true in $\mathfrak{K}$.

The formula $\mathfrak{F}_1 \vee \ldots \vee \mathfrak{F}_m$ is satisfiable in $\mathfrak{K}$ if and only if at least one of the $\mathfrak{F}_j$ is satisfiable in $\mathfrak{K}$.

Two closed formulas $\mathfrak{F}$ and $\mathfrak{F}_1$ of signature σ are said to be *equivalent*

in a class $\mathfrak{K}$ of algebraic systems of signature σ if the value of $\mathfrak{F}$ coincides with that of $\mathfrak{F}_1$ in each system belonging to $\mathfrak{K}$.

It is clear that two closed formulas $\mathfrak{F}$ and $\mathfrak{F}_1$ of signature σ are equivalent in a class $\mathfrak{K}$ of systems having signature σ if and only if the formula

$$(\mathfrak{F} \to \mathfrak{F}_1) \,\&\, (\mathfrak{F}_1 \to \mathfrak{F}) \tag{1}$$

is true in $\mathfrak{K}$.

When $\mathfrak{F}$ and $\mathfrak{F}_1$ are nonclosed formulas of signature σ, we shall say that they are *equivalent* in a class $\mathfrak{K}$ of systems of signature σ if (1) is a true formula in $\mathfrak{K}$.

Let $\mathfrak{S}$ be a family of closed APC-formulas of signature σ. The class of all algebraic systems of signature σ in each of which every formula belonging to $\mathfrak{S}$ is true will be denoted by $K\mathfrak{S}$. It is clear that if $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, then $K\mathfrak{S}_1 \supseteq K\mathfrak{S}_2$.

Let $\mathfrak{K}$ be an arbitrary class of algebraic systems of signature σ. The collection of all closed APC-formulas of signature σ that are true in $\mathfrak{K}$ is said to be the *elementary theory for* $\mathfrak{K}$ and is denoted by Th $\mathfrak{K}$. Observe that if $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$, then Th $\mathfrak{K}_1 \supseteq$ Th $\mathfrak{K}_2$. It is also clear that $\mathfrak{K} \subseteq K$ Th $\mathfrak{K}$.

A class $\mathfrak{K}$ of systems of signature σ is said to be *axiomatizable* if

$$\mathfrak{K} = K \text{ Th } \mathfrak{K}.$$

Theorem 1. *A class $\mathfrak{K}$ of algebraic systems of signature σ is axiomatizable if and only if there exists a collection $\mathfrak{S}$ of closed APC-formulas of signature σ such that $\mathfrak{K} = K\mathfrak{S}$.*

If $\mathfrak{K} = K$ Th $\mathfrak{K}$, then $\mathfrak{S}$ may be taken to be Th $\mathfrak{K}$. Suppose $\mathfrak{S}$ is a collection of closed APC-formulas of signature σ such that $\mathfrak{K} = K\mathfrak{S}$. Since $\mathfrak{S} \subseteq$ Th $\mathfrak{K}$, it follows that $K\mathfrak{S} \supseteq K$ Th $\mathfrak{K}$. On the other hand, $\mathfrak{K} \subseteq K$ Th $\mathfrak{K}$. Thus, $\mathfrak{K} = K$ Th $\mathfrak{K}$.

Problems and Complements

1. The set A_Ω of all axiomatizable classes of algebraic systems of fixed signature Ω has cardinality $2^{|\Omega|+\aleph_0}$.

2. Let $\mathfrak{A}$ be an algebraic system of signature Ω. The cardinality of a subsystem of $\mathfrak{A}$ generated by a set C cannot exceed $\max\{|\Omega|, |C|, \aleph_0\}$.

3. There exists an effective procedure for writing down in succession all identically true APC-formulas (Hilbert's algorithm; see Novikov [56], Kleene [28] and Hilbert and Ackermann [21]).

4. Let σ be a finite signature and let $\mathfrak{S}$ be a collection of quantifier-free formulas of signature σ each involving no free object variables other than $y_1, \ldots, y_r$. $\mathfrak{S}$ is said to be a *consistent* collection if there exists a system $\mathfrak{M}$ of signature σ from which it is possible to assign values to $y_1, \ldots, y_r$ so that every formula in $\mathfrak{S}$ is true in $\mathfrak{M}$ under these values. *Localization Principle* for quantifier-free APC-formulas:

*If each finite subcollection of a collection of formulas $\mathfrak{S}$ is consistent, then the whole collection $\mathfrak{S}$ is consistent.**)

Proof. Consider all APC-formulas of signature σ of the form

$$\mathfrak{a}_1 = \mathfrak{a}_2, \quad P^{(n)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n),$$

$\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ are terms all of whose operator symbols belong to σ and object variables belong to the set $\{y_1, \ldots, y_r\}$; $P^{(n)}$ is an n-ary predicate symbol in σ. Arrange them in a sequence $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n, \ldots$ and construct a collection of formulas $\mathfrak{S}_0, \mathfrak{S}_1, \ldots$ in succession as follows. Let $\mathfrak{S}_0 = \mathfrak{S}$. Let $\mathfrak{S}_{i+1} = \mathfrak{S}_i \cup \{\neg \mathfrak{F}_{i+1}\}$ if for some finite subsystem $\mathfrak{S}'$ in $\mathfrak{S}_i$, the collection of formulas $\mathfrak{S}' \cup \{\mathfrak{F}_{i+1}\}$ is inconsistent. Otherwise, let $\mathfrak{S}_{i+1} = \mathfrak{S}_i \cup \{\mathfrak{F}_{i+1}\}$. Set $\mathfrak{S}_\infty = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$. Denote by A the set of terms each of whose operator symbols belongs to σ and whose object variables belong to the set $\{y_1, \ldots, y_r\}$. For each pair $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in A, call $\mathfrak{a}_1 \sim \mathfrak{a}_2$ if $(\mathfrak{a}_1 = \mathfrak{a}_2) \in \mathfrak{S}_\infty$. The reader can verify that $\sim$ is an equivalence relation on A. To each $\mathfrak{a}$ in A, let $[\mathfrak{a}]$ denote the equivalence class containing $\mathfrak{a}$ and let $A/\sim$ be the set of all such equivalence classes for all possible $\mathfrak{a}$ in A. Define operations $f^{(n)}$ and predicates $P^{(m)}$ on the quotient set $A/\sim$ corresponding to all $f^{(n)}$ and $P^{(m)}$ in σ by setting

$$f^{(n)}([\mathfrak{a}_1], \ldots, [\mathfrak{a}_n]) = [f^{(n)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)];$$

$$P^{(m)}([\mathfrak{a}_1], \ldots, [\mathfrak{a}_m]) = \mathsf{T} \Leftrightarrow P^{(m)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_m) \in \mathfrak{S}_\infty.$$

It is easy to show that every formula in $\mathfrak{S}$ is true in the resulting system of signature σ for the values $[y_i]$ for y_i, $i = 1, \ldots, r$.

7. Classification of Formulas

7.1 ∀-Formulas and ∃-Formulas. It is known that every formula of signature σ in the applied predicate calculus is equivalent**) to a formula of signature σ in *prenex normal form*

$$(Q_1 x_1) \ldots (Q_m x_m)\, \mathfrak{A}(x_1, \ldots, x_m, y_1, \ldots, y_n).$$

Here $\mathfrak{A}$ is a quantifier-free formula and the symbols Q_i stand for $\forall$ or $\exists$ (see Novikov [56], Kleene [28] and Hilbert and Ackermann [21]). Formulas in prenex normal form***) are classified according to the composition of their *prefixes*, i.e., the quantifier portion $(Q_1 x_1) \ldots (Q_m x_m)$. Namely, one says that an APC-Formula of the form

$$(Q_1 x_{11} \ldots x_{1k_1}) \ldots (Q_l x_{l1} \ldots x_{lk_l})\, \mathfrak{A},$$

---

*) Sometimes called the *compactness theorem* (see [11]) [*Trans.*].

**) Second-order formulas $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are said to be equivalent if the formula $(\mathfrak{F}_1 \to \mathfrak{F}_2) \,\&\, (\mathfrak{F}_2 \to \mathfrak{F}_1)$ is identically true.

***) *Prenex formulas.*

*If each finite subcollection of a collection of formulas $\mathfrak{S}$ is consistent, then the whole collection $\mathfrak{S}$ is consistent.**)

Proof. Consider all APC-formulas of signature σ of the form

$$\mathfrak{a}_1 = \mathfrak{a}_2, \quad P^{(n)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n),$$

$\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ are terms all of whose operator symbols belong to σ and object variables belong to the set $\{y_1, \ldots, y_r\}$; $P^{(n)}$ is an n-ary predicate symbol in σ. Arrange them in a sequence $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n, \ldots$ and construct a collection of formulas $\mathfrak{S}_0, \mathfrak{S}_1, \ldots$ in succession as follows. Let $\mathfrak{S}_0 = \mathfrak{S}$. Let $\mathfrak{S}_{i+1} = \mathfrak{S}_i \cup \{\neg \mathfrak{F}_{i+1}\}$ if for some finite subsystem $\mathfrak{S}'$ in $\mathfrak{S}_i$, the collection of formulas $\mathfrak{S}' \cup \{\mathfrak{F}_{i+1}\}$ is inconsistent. Otherwise, let $\mathfrak{S}_{i+1} = \mathfrak{S}_i \cup \{\mathfrak{F}_{i+1}\}$. Set $\mathfrak{S}_\infty = \bigcup_{i=0}^{\infty} \mathfrak{S}_i$. Denote by A the set of terms each of whose operator symbols belongs to σ and whose object variables belong to the set $\{y_1, \ldots, y_r\}$. For each pair $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in A, call $\mathfrak{a}_1 \sim \mathfrak{a}_2$ if $(\mathfrak{a}_1 = \mathfrak{a}_2) \in \mathfrak{S}_\infty$. The reader can verify that $\sim$ is an equivalence relation on A. To each $\mathfrak{a}$ in A, let $[\mathfrak{a}]$ denote the equivalence class containing $\mathfrak{a}$ and let $A/\sim$ be the set of all such equivalence classes for all possible $\mathfrak{a}$ in A. Define operations $f^{(n)}$ and predicates $P^{(m)}$ on the quotient set $A/\sim$ corresponding to all $f^{(n)}$ and $P^{(m)}$ in σ by setting

$$f^{(n)}([\mathfrak{a}_1], \ldots, [\mathfrak{a}_n]) = [f^{(n)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)];$$

$$P^{(m)}([\mathfrak{a}_1], \ldots, [\mathfrak{a}_m]) = \mathsf{T} \Leftrightarrow P^{(m)}(\mathfrak{a}_1, \ldots, \mathfrak{a}_m) \in \mathfrak{S}_\infty.$$

It is easy to show that every formula in $\mathfrak{S}$ is true in the resulting system of signature σ for the values $[y_i]$ for y_i, $i = 1, \ldots, r$.

7. Classification of Formulas

7.1 ∀-Formulas and ∃-Formulas. It is known that every formula of signature σ in the applied predicate calculus is equivalent**) to a formula of signature σ in *prenex normal form*

$$(Q_1 x_1) \ldots (Q_m x_m)\, \mathfrak{A}(x_1, \ldots, x_m, y_1, \ldots, y_n).$$

Here $\mathfrak{A}$ is a quantifier-free formula and the symbols Q_i stand for $\forall$ or $\exists$ (see Novikov [56], Kleene [28] and Hilbert and Ackermann [21]). Formulas in prenex normal form***) are classified according to the composition of their *prefixes*, i.e., the quantifier portion $(Q_1 x_1) \ldots (Q_m x_m)$. Namely, one says that an APC-Formula of the form

$$(Q_1 x_{11} \ldots x_{1k_1}) \ldots (Q_l x_{l1} \ldots x_{lk_l})\, \mathfrak{A},$$

---

*) Sometimes called the *compactness theorem* (see [11]) [*Trans.*].

**) Second-order formulas $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are said to be equivalent if the formula $(\mathfrak{F}_1 \to \mathfrak{F}_2) \,\&\, (\mathfrak{F}_2 \to \mathfrak{F}_1)$ is identically true.

***) *Prenex formulas.*

in which the $Q_j x_{j1} \dots x_{jk_j}$ $(j = 1, \dots, l)$ are alternating quantifiers, is of type $Q_1 \dots Q_l$. For example, the formulas

$$(\forall xy)xy \geqq x + y,$$
$$(\forall x)\,(\exists yz)x = yz,$$
$$(\exists x)\,(\forall yz)\,(\exists uv)y \cdot u = z \cdot v$$

are respectively of type $\forall$, $\forall\exists$ and $\exists\forall\exists$.

A formula of type $Q_1 \dots Q_l$ is also called a $Q_1 \dots Q_l$-*formula*. The prefix type of a formula is not invariant under equivalence. There are equivalent formulas in prenex normal form of different prefix type. For instance, all identically true formulas are equivalent to each other. But there are identically true formulas of any given prefix type.

$\forall$-formulas are said to be *universal formulas* and $\exists$-formulas, *existential formulas*.

Theorem 1. *Let*

$$(\forall x_1 \dots x_m)\mathfrak{A}(x_1, \dots, x_m, y_1, \dots, y_n) \tag{1}$$

be an $\forall$*-formula of signature* σ*, in which* $\mathfrak{A}$ *is quantifier-free, which is true in an algebraic system* $\mathfrak{M}$ *of signature* σ *for* $y_1 = y_{10}, \dots, y_n = y_{n0}$ *in* $\mathfrak{M}$*. Then* (1) *is a true formula for these values in any subsystem* $\mathfrak{N}$ *of* $\mathfrak{M}$ *containing them.*

The fact that (1) is a true formula for $y_1 = y_{10}, \dots, y_n = y_{n0}$ means that the quantifier-free formula $\mathfrak{A}(x_1, \dots, x_m, y_{10}, \dots, y_{n0})$ is true for any values of the variables $x_1, \dots, x_m$ from $\mathfrak{M}$. But then such a quantifier-free formula is also true for all values $x_1, \dots, x_m$ in $\mathfrak{N}$. In other words, (1) is a true formula in $\mathfrak{N}$ at the point $y_{10}, \dots, y_{n0}$.

As already mentioned in Section 6.3, formula (1) defines a predicate $A_{\mathfrak{M}}(y_1, \dots, y_n)$ in each algebraic system $\mathfrak{M}$ of signature σ which depends on the whole structure of $\mathfrak{M}$. Theorem 1 asserts that the value of the predicate at a given point remains unchanged when going to a subsystem, i.e.,

$$A_{\mathfrak{M}}(y_1, \dots, y_n) = A_{\mathfrak{N}}(y_1, \dots, y_n)$$

for each subsystem $\mathfrak{N}$ of $\mathfrak{M}$ containing $y_1, \dots, y_n$. This invariance property is sometimes referred to as *stability under passage to a subsystem.*

For $n = 0$, Theorem 1 reduces to the following statement: *If a closed universal formula is true in an algebraic system, then it is true in any subsystem of it.*

Passing from $\forall$-formulas to their negations, we obtain a companion result.

Corollary 1. *Let*

$$(\exists x_1 \dots x_m)\,\mathfrak{A}(x_1, \dots, x_m, y_1, \dots, y_n), \tag{2}$$

be an $\exists$*-formula, in which* $\mathfrak{A}$ *is a quantifier-free formula of signature* σ *involving free variables* $x_1, \dots, x_m, y_1, \dots, y_n$*, which is true in any algebraic system* $\mathfrak{N}$ *of signature* σ *for* $y_1 = y_{10}, \dots, y_n = y_{n0}$*. Then* (2) *is also a true formula for these values in each system* $\mathfrak{M}$ *of which* $\mathfrak{N}$ *is a subsystem. In other words, each* $\exists$*-predicate is stable under passage to a supersystem.*

To see this, observe that the negation of (2) is equivalent to the $\forall$-formula

$$(\forall x_1 \dots x_m) \neg \mathfrak{A}(x_1, \dots, x_m, y_1, \dots, y_n). \tag{3}$$

If (2) were false in $\mathfrak{M}$ at the point $y_{10}, \dots, y_{n0}$, then the $\forall$-formula (3) would be true in $\mathfrak{M}$ for the values $y_{10}, \dots, y_{n0}$. By Theorem 1, (3) would be a true formula in $\mathfrak{N}$ for these values, i.e., (2) would be false in $\mathfrak{N}$, contradicting the hypothesis.

Corollary 2. *There are* $\forall$*-formulas* ($\exists$*-formulas*) *that are not equivalent to any* $\exists$*-formula* ($\forall$*-formula*).

As an example, consider the $\exists$-formula $(\exists x)P(x)$ and the model $\mathfrak{M} = \langle \{0, 1\}; P\rangle$, where $P(0) = \mathsf{F}$ and $P(1) = \mathsf{T}$. The formula is true in $\mathfrak{M}$ but false in the submodel $\langle \{0\}; P\rangle$. Therefore, $(\exists x)P(x)$ is not equivalent to any $\forall$-formula.

Predicates specified by quantifier-free formulas are clearly stable in either direction (passing to a subsystem or to a supersystem). Let us show that the only bilaterally stable $\forall$-formulas and $\exists$-formulas are those equivalent to quantifier-free formulas.

Theorem 2. *If the* $\exists$*-formula* (2) *is stable in passing to a subsystem, then it is equivalent to a quantifier-free formula.*

Let (2) be a true formula in an algebraic system $\mathfrak{M}$ of signature σ at the point $y_{10}, \dots, y_{n0}$. Take the subsystem $\mathfrak{N}$ generated in $\mathfrak{M}$ by $y_{10}, \dots, y_{n0}$. By hypothesis, formula (2) is true in $\mathfrak{N}$ at the point $y_{10}, \dots, y_{n0}$. Thus $\mathfrak{N}$ contains elements $x_1, \dots, x_m$ for which $\mathfrak{A}(x_1, \dots, x_m, y_{10}, \dots, y_{n0}) = \mathsf{T}$. But each element of $\mathfrak{N}$ is the value of some term in $y_{10}, \dots, y_{n0}$ (see Section 6.1). Hence, for appropriate terms $T_1^0, \dots, T_m^0$ in $y_{10}, \dots, y_{n0}$,

$$\mathfrak{A}(T_1^0, \dots, T_m^0, y_{10}, \dots, y_{n0}) \tag{4}$$

is a true formula.

Consider now the collection $\mathfrak{S}$ of formulas $\mathfrak{F}$ of the form (2) and all formulas of the form $\neg \mathfrak{A}(T_1, \dots, T_m, y_1, \dots, y_n)$, in which $T_1, \dots, T_m$ are any terms in $y_1, \dots, y_n$. The collection $\mathfrak{S}$ is not satisfiable. For, suppose $\mathfrak{S}$ were satisfied in an algebraic system $\mathfrak{M}$ of signature σ at the

point $y_{10}, \ldots, y_{n0}$. Then for certain terms $T_1, \ldots, T_m$, formula (4) would be true. But this is impossible since the formula $\neg \mathfrak{A}(T_1, \ldots, T_m, y_1, \ldots, y_n)$ belongs to $\mathfrak{S}$.

Thus the infinite collection $\mathfrak{S}$ is not satisfiable. By the localization principle (see Item 4 in the Problems and Complements to Section 6), it contains a finite subcollection

$$\mathfrak{F}, \neg \mathfrak{A}(T_{11}, \ldots, T_{1m}, y_1, \ldots, y_n), \ldots, \neg \mathfrak{A}(T_{k1}, \ldots, T_{km}, y_1, \ldots, y_n)$$

which is not satisfiable, i.e., such that the formula

$$\mathfrak{F} \,\&\, \neg \mathfrak{A}(T_{11}, \ldots, T_{1m}, y_1, \ldots, y_n) \,\&\, \ldots \,\&\, \neg \mathfrak{A}(T_{k1}, \ldots, T_{km}, y_1, \ldots, y_n)$$

is not satisfiable. And hence the formula

$$\mathfrak{F} \to \mathfrak{A}(T_{11}, \ldots, T_{1m}, y_1, \ldots, y_n) \vee \ldots \vee \mathfrak{A}(T_{k1}, \ldots, T_{km}, y_1, \ldots, y_n)$$

is identically true.

The converse implication,

$$\mathfrak{A}(T_{11}, \ldots, T_{1m}, y_1, \ldots, y_n) \vee \ldots \vee \mathfrak{A}(T_{k1}, \ldots, T_{km}, y_1, \ldots, y_n) \to \mathfrak{F}$$

is also identically true. For if the left-hand member of the implication is true in algebraic system $\mathfrak{M}$ for some $y_1, \ldots, y_n$, then some formula $\mathfrak{A}(T_{i1}, \ldots, T_{im}, y_1, \ldots, y_n)$ is true in $\mathfrak{M}$. Therefore $\mathfrak{A}(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a true formula for $x_1 = T_{i1}, \ldots, x_m = T_{im}$ and so $\mathfrak{F}$ is also true. Thus $\mathfrak{F}$ is equivalent to the quantifier-free formula

$$\mathfrak{A}(T_{11}, \ldots, T_{1m}, y_1, \ldots, y_n) \vee \ldots \vee \mathfrak{A}(T_{k1}, \ldots, T_{km}, y_1, \ldots, y_n), \qquad (5)$$

as asserted.

The proof of Theorem 2 is not constructive in that it does not point out a procedure for obtaining the formula (5). However such an algorithm does exist. By use of Hilbert's algorithm (see Item 3 in the Problems and Complements to Section 6), we construct all identically true formulas of the form $\mathfrak{F} \to \mathfrak{F}_1$, $\mathfrak{F} \to \mathfrak{F}_2, \ldots$ At the same time, we construct the chain of all identically true formulas of the form $\mathfrak{F}_i \to \mathfrak{F}_{i1}, \mathfrak{F}_i \to \mathfrak{F}_{i2}, \ldots$ and we continue the process until we obtain the formulas $\mathfrak{F} \to \mathfrak{F}_i$, $\mathfrak{F}_i \to \mathfrak{F}$, where $\mathfrak{F}_i$ is a quantifier-free formula. Theorem 2 guarantees that such an $\mathfrak{F}_i$ exists.

By considering the negation of an $\forall$-formula, we have as a direct consequence of Theorem 2 the following companion result.

Corollary. *If an $\forall$-formula is stable in passing to a supersystem, then it is equivalent to a quantifier-free formula.*

We now pose the following question. Does there exist a decision procedure for determining whether or not an APC-formula $\mathfrak{F}$ is equivalent to an $\forall$-formula? The following holds.

Theorem 3. *No decision procedure exists for determining from the expression for a closed APC-formula $\mathfrak{F}$ whether or not $\mathfrak{F}$ is equivalent to an $\exists$-formula ($\forall$-formula).*

By Church's theorem (see Kleene [28], p. 432, Theorem 54), there is no decision procedure for determining whether a RPC-formula is identically true. We shall show that if $\mathscr{A}$ were a decision procedure for determining whether a formula is equivalent to some $\exists$-formula, then $\mathscr{A}$ could be used to determine identically true formulas, which would contradict Church's theorem.

Thus suppose we have the stipulated decision procedure and suppose $\mathfrak{F}$ is any RPC-formula. Let us see whether or not $\mathfrak{F}$ is equivalent to an $\exists$-formula. If not, then $\mathfrak{F}$ cannot be identically true since each identically true formula is equivalent to the $\exists$-formula $(\exists x)x = x$. Suppose the answer is 'yes'. Then by means of Hilbert's procedure, we can construct in succession all formulas equivalent to $\mathfrak{F}$ and from among these we can select the first $\exists$-formula $\mathfrak{F}_1$. It remains to ascertain whether or not $\mathfrak{F}_1$ is identically true, i.e., whether or not $\neg\mathfrak{F}_1$ is satisfiable. The formula $\neg\mathfrak{F}_1$ can be reduced to universal form in the usual way. Hence, by Theorem 1 the question of whether $\neg\mathfrak{F}_1$ is satisfiable is equivalent to the question of whether $\neg\mathfrak{F}_1$ is satisfiable in some one-element system. Thus the first question is effectively decidable.

The result stated in Theorem 3 can be proved for an $\forall$-formula in a similar way.

We conclude this section with some remarks apropos of Theorem 2. Formulas such as

$$P(\mathfrak{x}_1, \ldots, \mathfrak{x}_m), \quad F(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) = \mathfrak{x}_{n+1},$$

in which the $\mathfrak{x}_i$ are object variables and P and F are signature predicate and operator symbols, respectively, are called *atomic*. Formulas derived from atomic formulas and the logical symbols &, $\vee$, $\neg$, $\rightarrow$ (i.e., without using quantifiers) are said to be *absolutely quantifier-free* or *open*. Formulas of the form

$$(Q_1 x_{11} \ldots x_{1r_1}) \ldots (Q_l x_{l1} \ldots x_{lr_l})\mathfrak{B},$$

in which $\mathfrak{B}$ is an open formula and $Q_1 x_{11} \ldots x_{1r_1}, \ldots, Q_l x_{l1} \ldots x_{lr_l}$ are alternating quantifiers are said to be *absolute $Q_1 \ldots Q_l$-formulas*.

If the signature contains no operator symbols, then the concepts of $Q_1 \dots Q_l$-formula and absolute $Q_1 \dots Q_l$-formula coincide. For an arbitrary signature, they do not coincide but the following proposition does hold.

Theorem 4. *Each $Q_1 \dots Q_l$-formula ($l \geqq 1$) is equivalent to some absolute $Q_1 \dots Q_l$-formula.*

To prove this, it suffices to show that a formula of the form $f = g$, where f and g are terms, is equivalent to both an absolute $\exists$-formula of the form

$$(\exists x_1 \dots x_p)\,(B_1 \,\&\, \dots \,\&\, B_q), \tag{6}$$

and an absolute $\forall$-formula of the form

$$(\forall x_1 \dots x_r)\,(C_1 \,\&\, \dots \,\&\, C_s \to C_0), \tag{7}$$

in which the B_i and C_j are atomic formulas.

The proof is by induction on the number n of operator symbols occurring in the formula $f = g$. For $n \leqq 1$, $f = g$ is an atomic formula and so there is nothing to prove. Suppose $n > 1$. Then $f = g$ is a formula of the form $F_i(f_1, \dots, f_m) = h$ and so is equivalent to

$$(\exists x_1 \dots x_{m+1})\,(x_1 = f_1 \,\&\, \dots \,\&\, x_m = f_m \,\&\, x_{m+1} = F_i(x_1, \dots, x_m) \,\&\, x_{m+1} = h). \tag{8}$$

Here F_i is a signature operator symbol and $x_1, \dots x_{m+1}$ are symbols not occurring in the terms f and g. The number of operator symbols in each of the subformulas

$$x_{m+1} = h, \qquad x_\lambda = f_\lambda \qquad (\lambda = 1, \dots, m)$$

is less than n. Therefore each is equivalent to a formula of the form (6). If we replace them in (8) by their equivalents and we then take out the existential quantifiers, we arrive at a formula of the form (6) equivalent to $f = g$.

To reduce formula $f = g$ to the form (7), we observe first that $f = g$ is equivalent to

$$(\forall x_1 \dots x_{m+1})\,(x_1 = f_1 \,\&\, \dots \,\&\, x_m = f_m \,\&\, x_{m+1} = F_i(x_1, \dots, x_m) \to x_{m+1} = h).$$

On replacing the subformulas $x_\lambda = f_\lambda$ $(\lambda = 1, \dots, m)$ by the formulas of the form (6) equivalent to them and subformula $x_{m+1} = h$ by the formula of the form (7) equivalent to it and then taking out the

quantifiers, we wind up with a formula of the form (7) equivalent to $f = g$. The proof of Theorem 4 is complete.

Thus the term equation $f = g$ can be reduced both to absolute $\exists$-form and to absolute $\forall$-form. A question is whether it can be reduced to absolute quantifier-free form. The answer is in the negative. As an example, consider the formula

$$F\big(x, F(x, x)\big) = x. \tag{9}$$

Suppose that it is equivalent to some absolute quantifier-free formula $\Phi(x, y, \ldots, z)$. Apart from the operator symbol F, this last formula may involve other operator symbols F_i as well as predicate symbols P_j. Since the formulas $F\big(x, F(x, x)\big) = x$ and $\Phi(x, y, \ldots, z)$ must have the same values for any values of $x, \ldots z$, F_i and P_j, there is no violation of the equivalence if we set $y = \cdots = z = x$, $F_i(x, x, \ldots) = x$ and $P_j(x, x, \ldots) = \mathsf{T}$ in Φ. This yields an equivalence of the form

$$F\big(x, F(x, x)\big) = \Psi(x),$$

in which $\Psi(x)$ is built up from the atomic formulas $x = x$ and $F(x, x) = x$ by means of the symbols &, $\vee$ and $\neg$. But every formula Ψ of this form is equivalent to one of the formulas

$$x = x, \quad x \neq x, \quad x = F(x, x), \quad x \neq F(x, x),$$

and none of these is equivalent to (9).

7.2 Universal Axiomatic Subclasses. On account of the great logical simplicity of assertions expressible through the use of $\forall$-formulas, it is also not difficult to describe the properties of algebraic system that can be defined by a collection of $\forall$-formulas (see Section 7.1.). We first introduce a few definitions.

Let $\mathfrak{A} = \langle A, \Omega \rangle$ be an algebraic system. If we replace the operator symbols in its signature Ω by corresponding predicate symbols (see Section 2.2), we obtain a new signature Ω_M. As a result, the algebraic system $\langle A, \Omega \rangle$ is converted into a model $\mathfrak{A}_M = \langle A, \Omega_M \rangle$. All possible submodels of $\mathfrak{A}_M$ are said to be submodels of the algebraic system $\mathfrak{A}$ (see Section 2.3). It is important to remember that though every subsystem of $\mathfrak{A}$ is a submodel not every submodel of $\mathfrak{A}$ is a subsystem (Section 2. 3).

If Ω' is a nonempty subset of the signature Ω, then the algebraic system $\langle A, \Omega' \rangle$ is called an *Ω'-reduct* (sometimes Ω'-projection or Ω'-restriction) of $\mathfrak{A} = \langle A, \Omega \rangle$. If the signature Ω' is finite, then an Ω'-reduct is called a *finite reduct* of $\mathfrak{A}$. For example, let

$$\mathfrak{A} = \langle R; \sqrt[2]{\ }, \sqrt[3]{\ }, \sqrt[4]{\ }, \ldots \rangle,$$

where R is the set of real numbers and $\sqrt[n]{|\ |}$ is the unary operation of extracting the n-th root of the absolute value of a number. Then the algebras

$$\mathfrak{A}_i = \langle R; \sqrt[2]{|\ |}, \sqrt[3]{|\ |}, \ldots, \sqrt[i]{|\ |} \rangle$$

are finite reducts of $\mathfrak{A}$.

Finite reducts of finite submodels of an algebraic system are called *local submodels* of the system.

Closely related to each submodel of an algebraic system is the concept of diagram of a model. Let $\mathfrak{A} = \langle A, \Omega \rangle$ be a model. With each element a in A, we associate a special symbol z_a not belonging to A or Ω. Consider all possible formulas of the form

$$P(z_{a_1}, \ldots, z_{a_n}), \quad \neg P(z_{a_1}, \ldots, z_{a_n}) \quad (a_i \in A), \tag{1}$$

where P is either the equality relation or a predicate symbol in Ω. The set of formulas of the form (1) for which the corresponding expression $P(a_1, \ldots, a_n)$ or $\neg P(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$ is called the *diagram of model* $\mathfrak{A}$ and is denoted by $D(\mathfrak{A})$. For example, the diagram of $\langle \{0, 1, 2\}; \leqq \rangle$ consists of the formulas

$$z_0 = z_0, \quad z_1 = z_1, \quad z_2 = z_2, \quad z_0 \neq z_1, \quad z_1 \neq z_0, \quad z_0 \neq z_2,$$
$$z_2 \neq z_0, \quad z_1 \neq z_2, \quad z_2 \neq z_1, \quad z_0 \leqq z_0, \quad z_1 \leqq z_1, \quad z_2 \leqq z_2,$$
$$z_0 \leqq z_1, \quad \neg z_1 \leqq z_0, \quad z_0 \leqq z_2, \quad \neg z_2 \leqq z_0, \quad z_1 \leqq z_2, \quad \neg z_2 \leqq z_1.$$

If the signature of model $\mathfrak{A}$ contains object variables c_i and $c_i = a_i$ in $\mathfrak{A}$ $(a_i \in A)$, then apart from the formulas (1), the diagram includes the formulas

$$c_i = z_{a_i}, \qquad c_i \neq z_a \qquad (a \neq a_i,\ a \in A).$$

Finally, if $\mathfrak{A}$ is not a model but an algebraic system whose signature Ω contains the operator symbols F_i, then the diagram of $\mathfrak{A}$ is defined to be the diagram of its corresponding model $\mathfrak{A}_M$. In other words, the diagram of $\mathfrak{A}$ now comprises formulas of the form (1), with P a predicate symbol in Ω, and formulas of the form

$$F(z_{a_1}, \ldots, z_{a_m}) = z_{a_0} \quad \big(a_0 = F(a_1, \ldots, a_m)\big),$$
$$F(z_{a_1}, \ldots, z_{a_m}) \neq z_{a_0} \quad \big(a_0 \neq F(a_1, \ldots, a_m)\big)$$
$$(a_0, \ldots, a_m \in A, \quad F \in \Omega).$$

For example, the diagram of the reduced submodel $\langle\{1, 2\}; \cdot\rangle$ of the ring $\langle\{0, \pm 1, \pm 2, \ldots\}; +, \cdot\rangle$ consists of the formulas

$$z_1 = z_1, \quad z_2 = z_2, \quad z_1 \neq z_2, \quad z_2 \neq z_1, \quad z_1 \cdot z_1 = z_1,$$
$$z_1 \cdot z_2 = z_2, \quad z_2 \cdot z_1 = z_2, \quad z_1 \cdot z_2 \neq z_1, \quad z_1 \cdot z_1 \neq z_2,$$
$$z_2 \cdot z_1 \neq z_1, \quad z_2 \cdot z_2 \neq z_1, \quad z_2 \cdot z_2 \neq z_2.$$

Let $\mathfrak{A}_{\alpha\beta} = \langle A_\alpha, \Omega_\beta\rangle$ be a finite reduct of a finite submodel of an algebraic system $\mathfrak{A} = \langle A, \Omega\rangle$. Then the diagram $D(\mathfrak{A}_{\alpha\beta})$ will involve a finite number of formulas. Let $D^\wedge(\mathfrak{A}_{\alpha\beta})$ denote the conjunction of all formulas in $D(\mathfrak{A}_{\alpha\beta})$ and let $(\exists_\delta) D^\wedge(\mathfrak{A}_{\alpha\beta})$ be an abbreviation for the formula

$$(\exists z_{a_1} \ldots z_{a_s}) D^\wedge(\mathfrak{A}_{\alpha\beta}), \tag{2}$$

where $a_1, \ldots, a_s$ are all elements of $\mathfrak{A}_{\alpha\beta}$. It is clear that *formula* (2) *is true in an algebraic system* $\mathfrak{B} = \langle B, \Omega'\rangle$ *if and only if* $\mathfrak{B}$ *contains a finite submodel* $\langle B_\alpha, \Omega'\rangle$ *such that its* Ω_β*-reduct* $(\Omega_\beta \subseteq \Omega')$ *is isomorphic to the model* $\mathfrak{A}_{\alpha\beta} = \langle A_\beta, \Omega_\beta\rangle$, *i.e., if and only if* $\mathfrak{A}_{\alpha\beta}$ *is embeddable in model* $\mathfrak{B}$.

This in turn implies that *the formula*

$$(\forall z_{a_1} \ldots z_{a_s}) \neg D^\wedge(\mathfrak{A}_{\alpha\beta}), \tag{3}$$

equivalent to the negation of (2), *is true in a class* $\mathfrak{K}$ *of algebraic systems of signature* Ω' $(\Omega' \supseteq \Omega_\beta)$ *if and only if the model* $\mathfrak{A}_{\alpha\beta}$ *is not embeddable in any system belonging to* $\mathfrak{K}$.

At his point, we introduce a few further definitions. We shall say that an algebraic system $\mathfrak{A} = \langle A, \Omega\rangle$ is *locally embeddable* in system $\mathfrak{B} = \langle B, \Omega_1\rangle$, $(\Omega_1 \supseteq \Omega)$ if each finite reduct $\langle A_\alpha, \Omega_\beta\rangle$ of each finite submodel of $\mathfrak{A}$ is embeddable in $\mathfrak{B}$, i.e., is isomorphic to an Ω_β-reduct of an appropriate finite submodel of $\mathfrak{B}$.

An algebraic system $\mathfrak{A}$ is *locally embeddable in a class* $\mathfrak{K}$ *of systems* if each finite reduct of each finite submodel of $\mathfrak{A}$ is embeddable in an appropriate system in $\mathfrak{K}$ (depending on the choice of finite reduct).

A subclass $\mathfrak{L}$ of a class $\mathfrak{K}$ of algebraic systems is *locally closed in* $\mathfrak{K}$ if from the fact that an arbitrary $\mathfrak{K}$-system is locally embeddable in $\mathfrak{L}$ it follows that the $\mathfrak{K}$-system belongs to $\mathfrak{L}$. A class $\mathfrak{L}$ of algebraic systems of signature Ω is said to be simply locally closed if $\mathfrak{L}$ is locally closed in the class of all algebraic systems of signature Ω.

We once more return to the consideration of $\forall$-formulas. A subclass $\mathfrak{L}$ of a class $\mathfrak{K}$ of algebraic systems is said to be *universally axiomatizable* within $\mathfrak{K}$ (or an $\forall$-*subclass* of $\mathfrak{K}$) if there is a collection $\mathfrak{S}$ of closed $\forall$-formulas of signature Ω such that $\mathfrak{L}$ consists of exactly those $\mathfrak{K}$-systems in which all formulas belonging to $\mathfrak{S}$ are true.

Theorem 1 (Tarski [70], Łos [34]). *A subclass* $\mathfrak{L}$ *of a class* $\mathfrak{K}$ *of algrabraic systems of signature* Ω *is universally axiomatizable within* $\mathfrak{K}$ *if and only if* $\mathfrak{L}$ *is locally closed in* $\mathfrak{K}$.

Necessity. Suppose $\mathfrak{L}$ consists of $\mathfrak{K}$-systems satisfying a set $\mathfrak{S}$ of $\forall$-formulas of given signature Ω and suppose $\mathfrak{A}$ is a given $\mathfrak{K}$-system locally embeddable in $\mathfrak{L}$. We have to prove that each formula Φ in $\mathfrak{S}$ is true in $\mathfrak{A}$. By assumption, Φ has the form

$$(\forall x_1 \ldots x_n)\mathscr{A}(x_1, \ldots, x_n). \tag{4}$$

Only a finite number of symbols from Ω occur in (4). Let these symbols be $P_1, \ldots, P_s$. Choose any elements $a_1, \ldots, a_n$ in $\mathfrak{A}$. We have to show that the expression $\mathscr{A}(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$. Consider the local submodel

$$\mathfrak{A}_{\alpha\beta} = \langle \{a_1, \ldots, a_n\}, \{P_1, \ldots, P_s\} \rangle$$

of $\mathfrak{A}$. By assumption, there exists an isomorphism σ from $\mathfrak{A}_{\alpha\beta}$ into an $\mathfrak{L}$-system $\mathfrak{B}$. The formula (4) is true in $\mathfrak{B}$ and hence*)

$$\mathscr{A}(a_1\sigma, \ldots, a_n\sigma) \underset{\mathfrak{B}}{=} \mathsf{T}.$$

By Section 6.3,

$$\mathscr{A}(a_1, \ldots, a_n) \underset{\mathfrak{A}_{\alpha\beta}}{=} \mathsf{T} \Leftrightarrow \mathscr{A}(a_1\sigma, \ldots, a_n\sigma) \underset{\mathfrak{B}}{=} \mathsf{T}$$

and so Φ is true in $\mathfrak{A}$.

Sufficiency. Let $\mathfrak{L}$ be a locally closed subclass of $\mathfrak{K}$. Let $\mathfrak{S}$ be the set of all closed $\forall$-formulas of signature Ω that are true in each $\mathfrak{L}$-system. It is necessary to show that each $\mathfrak{K}$-system $\mathfrak{A}$ in which all of the formulas of $\mathfrak{S}$ are true is locally embeddable in $\mathfrak{L}$ (and therefore belongs to $\mathfrak{L}$). Suppose to the contrary that $\mathfrak{A}$ contains a local submodel $\mathfrak{A}_{\alpha\beta} = \langle A_\alpha, \Omega_\beta \rangle$ which is not embeddable in any $\mathfrak{L}$-system. By the remark made above, this means that formula (3) is true in all $\mathfrak{L}$-systems. But (3) is an $\forall$-formula of signature Ω and so belongs to $\mathfrak{S}$. Hence, (3) is true in system $\mathfrak{A}$, which means in particular that $\neg D^\wedge(\mathfrak{A}_{\alpha\beta})$ is true in $\mathfrak{A}$ for $z_{a_1} = a_1, \ldots, z_{a_n} = a_n$, where $A_\alpha = \{a_1, \ldots, a_n\}$. But from the definition of diagram, it is apparent $D^\wedge(\mathfrak{A}_{\alpha\beta})$ is true in $\mathfrak{A}$ for these values of $z_{a_1}, \ldots, z_{a_n}$. Thus, resulting is the impossible fact that both $D^\wedge(\mathfrak{A}_{\alpha\beta})$ and its negation are true in $\mathfrak{A}$ for $z_{a_1} = a_1, \ldots, z_{a_n} = a_n$.

Let $\mathfrak{K}$ be a class of systems of signature Ω. Then $L\mathfrak{K}$ denotes the class of all *models* of corresponding signature Ω_M locally embeddable

*) If $\mathfrak{A}$ and $\mathscr{A}$ are, respectively, any algebraic system and formula, the notation $\mathscr{A} \underset{\mathfrak{A}}{=} \mathsf{T}$ will mean that $\mathscr{A}$ is true in $\mathfrak{A}$ [*Ed.*].

in $\mathfrak{K}$. If $\mathfrak{K}_1$ is any other class of systems of signature Ω, then $\mathfrak{K} \cap L\mathfrak{K}_1$ is the class of all $\mathfrak{K}$-systems that are locally embeddable in $\mathfrak{K}_1$. From the definition of local embedding it is immediately apparent that

$$LL\mathfrak{K}_1 = L\mathfrak{K}_1. \tag{5}$$

Theorem 1 and relation (5) imply the following important result.

Corollary 2. *Let $\mathfrak{K}$ and $\mathfrak{K}_1$ be arbitrary classes of algebraic systems each of the same signature. Then the subclass $\mathfrak{L}$ of $\mathfrak{K}$-systems which are locally embeddable in $\mathfrak{K}_1$ is universally axiomatizable within $\mathfrak{K}$.*

For, if $\mathfrak{A}$ is any $\mathfrak{K}$-system locally embeddable in $\mathfrak{L}$, then by (5) $\mathfrak{A}$ is also locally embeddable in $\mathfrak{K}_1$. Hence it follows that $\mathfrak{A} \in \mathfrak{L}$ and so the subclass $\mathfrak{L}$ is locally closed in $\mathfrak{K}$.

A subclass $\mathfrak{L}$ of a class $\mathfrak{K}$ of algebraic systems of signature Ω is said to be *hereditary in* $\mathfrak{K}$ if each $\mathfrak{K}$-subsystem of any $\mathfrak{L}$-system is an $\mathfrak{L}$-system. The class $\mathfrak{L}$ is (*absolutely*) *hereditary* if it hereditary in the class of all algebraic systems of given signature, i.e., if every subsystem of an $\mathfrak{L}$-system is an $\mathfrak{L}$-system.

Every locally closed subclass $\mathfrak{L}$ of class $\mathfrak{K}$ is clearly hereditary in $\mathfrak{K}$ and so Theorem 1 immediately implies the next result.

Corollary 3. *If subclass $\mathfrak{L}$ is universally axiomatizable in $\mathfrak{K}$, then $\mathfrak{L}$ is hereditary in $\mathfrak{K}$.*

However, examples easily show that the converse does not hold.

Theorem 1 talks about the axiomatizability of a subclass by means of some generally infinite collection of $\forall$-formulas. Some obvious changes in the definitions easily lead to a corresponding theorem for $\mathfrak{L}$ to be axiomatizable by a *finite* collection of $\forall$-formulas.

$\forall$-formulas of the form (4) having the model signature Ω_M and containing n given quantifiers are called $\forall^n$-*formulas*. If the signature Ω is finite, the number of nonequivalent closed $\forall^n$-formulas is also finite. Therefore axiomatizing a subclass of finite signature by means of a finite number of $\forall$-formulas is equivalent to axiomatizing the subclass by a family of $\forall^n$-formulas for some fixed n.

An algebraic system $\mathfrak{A}$ is said to be $\forall^n$-*embeddable* in class $\mathfrak{L}$ if every finite reduct of every (at most) n-element submodel of $\mathfrak{A}$ is embeddable in some $\mathfrak{L}$-system. The subclass $\mathfrak{L}$ of $\mathfrak{K}$ is said to be $\forall^n$-*closed* in $\mathfrak{K}$ if every $\mathfrak{K}$-system $\forall^n$-embeddable in $\mathfrak{L}$ belongs to $\mathfrak{L}$.

Theorem 4. *A necessary and sufficient condition for a subclass $\mathfrak{L}$ of $\mathfrak{K}$ to be $\forall^n$-axiomatizable in $\mathfrak{K}$ is that $\mathfrak{L}$ be $\forall^n$-closed in $\mathfrak{K}$.*

The proof is similar to that of Theorem 1.

7.3 $\forall\exists$-Formulas and $\exists\forall$-Formulas. As already stated, formulas of the form

$$(\forall x_1 \ldots x_m)(\exists y_1 \ldots y_n)\mathfrak{A}(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p), \tag{1}$$

where $\mathfrak{A}$ is quantifier-free and x_i, y_j and z_k are the free object variables in $\mathfrak{A}$, are called $\forall\exists$-formulas. Such formulas are also referred to as *Skolem formulas* or formulas in Skolem normal form.

The negation of a Skolem formula is equivalent to a formula of the form

$$(\exists x_1 \ldots x_m)(\forall y_1 \ldots y_n)\mathfrak{B}(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p), \tag{2}$$

or in other words, to an $\exists\forall$-formula. And conversely, the negation of an $\exists\forall$-formula is equivalent to an $\forall\exists$-formula.

An appropriate question is: Do the Skolem formulas possess some kind of "stability" analogous to that of the $\forall$-formulas and $\exists$-formulas? The basic structural property of $\exists\forall$-formulas is given by the following theorem.

Theorem 1. *Suppose the $\exists\forall$-formula* (2) *is true in an algebraic system $\mathfrak{M}$ for certain values of $z_1, \ldots, z_p$ in $\mathfrak{M}$ and suppose $\mathfrak{M}$ can be covered by a local family of its subsystems $\mathfrak{M}_\alpha$ (for the definition, see Section 2.3). Then* (2) *is a true formula in any $\mathfrak{M}_\alpha$ containing the elements $z_1, \ldots, z_p$.*

The fact that (2) is true in $\mathfrak{M}$ means that for some elements $x_1, \ldots, x_m$ in $\mathfrak{M}$, the formula $(\forall y_1 \ldots y_n)\mathfrak{B}(x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p)$ is true. From the local character of the cover $\{\mathfrak{M}_\alpha\}$, there is a subsystem $\mathfrak{M}_\alpha$ belonging to the cover containing $x_1, \ldots, x_m, z_1, \ldots, z_p$. But the indicated formula is true in $\mathfrak{M}_\alpha$ for these elements. Therefore (2) is also true in $\mathfrak{M}_\alpha$ for the given values of the variables $z_1, \ldots, z_p$.

For $p = 0$,Theorem 1 yields: *If an algebraic system $\mathfrak{M}$ is locally covered by its subsystems $\mathfrak{M}_\alpha$ and a closed $\exists\forall$-formula $\mathfrak{F}$ is true in $\mathfrak{M}$, then $\mathfrak{F}$ is true* in *at least one subsystem of the cover.*

A simple passage from $\exists\forall$-formulas to their negations yields the following.

Corollary 1. *Suppose an algebraic system $\mathfrak{M}$ is covered by a local family of its subsystems $\mathfrak{M}_\alpha$ and suppose the $\forall\exists$-formula* (1) *is true for certain values of $z_1, \ldots, z_p$ in each subsystem $\mathfrak{M}_\alpha$ containing $z_1, \ldots, z_p$. Then* (1) *is also a true formula in $\mathfrak{M}$ for these values of $z_1, \ldots, z_p$.*

Suppose to the contrary that (1) were false on $\mathfrak{M}$. Then its negation, being of the form (2), would be true on $\mathfrak{M}$. Hence, by Theorem 1, it would have to be true in at least one subsystem $\mathfrak{M}_\alpha$ containing $z_1, \ldots, z_p$, thus contradicting the hypothesis.

For $p = 0$, Corollary 1 yields the result: *If an algebraic system* $\mathfrak{M}$ *is the union of an ascending chain of its subsystems,*

$$\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}_k \subset \cdots,$$

and some closed $\forall\exists$*-formula* $\mathfrak{F}$ *is true in each of these subsystems, then* $\mathfrak{F}$ *is also true in* $\mathfrak{M}$.

Indeed, the family $\{\mathfrak{M}_i\}$ is a local cover for $\mathfrak{M}$ and so Corollary 1 is applicable to it.

Corollary 2. *There exists an* $\exists\forall$*-formula* ($\forall\exists$*-formula*) *equivalent to no* $\forall\exists$*-formula* ($\exists\forall$*-formula*.)

Consider the $\exists\forall$-formula $(\exists x)(\forall y) y \leqq x$, which says that in a model with relation $\leqq$ there is a largest element x. Consider the model $\mathfrak{M} = \langle N, \leqq\rangle$, where $N = \{0, 1, \ldots\}$ and $\leqq$ is the customary order relation. Its submodels

$$\mathfrak{M}_i = \langle\{0, 1, \ldots, i\}; \leqq\rangle \quad (i = 1, 2, \ldots)$$

form an ascending chain whose union is $\mathfrak{M}$. The formula $(\exists x)(\forall y) y \leqq x$ is true in each submodel $\mathfrak{M}_i$ but false in the model $\mathfrak{M}$. Therefore $(\exists x)(\forall y) y \leqq x$ is equivalent to no $\forall\exists$-formula. Its negation $(\forall x)(\exists y)\neg y \leqq x$ is for the same reason not equivalent to any $\exists\forall$-formula.

In the preceding subsection, it was shown that if a formula $\mathfrak{F}$ is simultaneously equivalent to an $\forall$-formula and an $\exists$-formula, then it is equivalent to a formula of lower quantifier-free form. The analogous assertion is false for $\forall\exists$- and $\exists\forall$-formulas. There are formulas simultaneously equivalent to some $\forall\exists$-formula and some $\exists\forall$-formula but at the same time not equivalent to any $\forall$-formula or $\exists$-formula.

An example of this is the formula

$$(\forall x) P(x) \,\&\, (\exists y) Q(y). \tag{3}$$

It is equivalent to each of the following two formulas:

$$(\forall x)(\exists y)\big(P(x) \,\&\, Q(y)\big), \quad (\exists y)(\forall x)\big(P(x) \,\&\, Q(y)\big).$$

At the same time, it is easy to see that (3) is unstable under passage to subsystems or supersystems and so is not equivalent to any $\forall$-formula or $\exists$-formula.

The structural properties of $\forall\exists\forall$-formulas, $\exists\forall\exists$-formulas and formulas with more complex prefixes are known but to formulate them is quite complicated. By means of them and other considerations as well,

it is not hard to deduce, for example, that the formula

$$(\forall x_1)(\exists y_1)\dots(\forall x_n)(\exists y_n)P(x_1, y_1, \dots, x_n, y_n)$$

is equivalent to no formula containing a fewer number of quantifiers.

Theorem 2 of Section 7.1 carries over to $\forall\exists$-formulas and formulas involving other prefixes as follows.

Set $\neg\forall = \exists$ and $\neg\exists = \forall$. The prefix $\neg Q_1 \neg Q_2 \dots \neg Q_l$, where $Q_i = \forall$ or $\exists$, is called the *dual prefix* of $Q_1 Q_2 \dots Q_l$.

Theorem 2. *If formula $\mathfrak{F}$ is equivalent to a formula with prefix $Q_1 \dots Q_l$ and to a formula with the dual prefix $\neg Q_1 \dots \neg Q_l$, then $\mathfrak{F}$ is equivalent to the conjunction of disjunctions of suitable formulas with shorter prefixes.*

We shall omit the proof here.

Theorem 3. *To any fixed prefix $Q_1 \dots Q_l$, there exists no decision procedure for determining from its expression whether a closed formula is equivalent to some $Q_1 \dots Q_l$-formula.*

The theorem was proved in Section 7.1 for $l = 1$. We shall not attempt to establish it in its general form here. Instead, we shall prove it merely for the prefix $\forall\exists$. As in Section 7.1, we are going to show that if there were some decision procedure $\mathscr{A}$ allowing us to tell whether or not a closed formula $\mathfrak{F}$ is equivalent to an $\forall\exists$-formula, then with the help of $\mathscr{A}$ we would be able to indicate if an arbitrarily given closed formula $\mathfrak{F}$ not involving any operator symbols is identically true or not, in contradiction to Church's theorem. Indeed, if $\mathfrak{F}$ turned out to be nonequivalent to any $\forall\exists$-formula, then $\mathfrak{F}$ would not be identically true since every identically true formula is equivalent to the $\forall\exists$-formula

$$(\forall x)(\exists y)\, x = y.$$

Suppose $\mathfrak{F}$ is equivalent to some $\forall\exists$-formula. Using Hilbert's algorithm, we can find a specific $\forall\exists$-formula

$$(\forall x_1 \dots x_m)(\exists y_1 \dots y_n)\mathfrak{A} \tag{4}$$

equivalent to $\mathfrak{F}$. As in Section 7.1, we may assume that (4) has the same signature as $\mathfrak{F}$ and in particular, that (4) involves no operator symbols. We need to find out whether (4) is identically true, or in other words, whether the formula

$$(\exists x_1 \dots x_m)(\forall y_1 \dots y_n) \neg \mathfrak{A}(x_1, \dots, x_m, y_1, \dots, y_n) \tag{5}$$

is satisfiable.

If (5) were true in a model $\mathfrak{M}$, then $\mathfrak{M}$ would have elements $a_1, \ldots, a_m$ such that the expression

$$\neg \mathfrak{A}(a_1, \ldots, a_m, y_1, \ldots, y_n) \tag{6}$$

is true for all $y_1, \ldots, y_n$ in $\mathfrak{M}$. Hence (6) would be true for all $y_1, \ldots, y_n$ in the submodel formed in $\mathfrak{M}$ by the elements $a_1, \ldots, a_m$.

Thus, if (5) were satisfiable, it would be satisfiable in a model containing at most m elements. But one can effectively construct all such models and so find out for each one if (5) is true or false in it.

7.4 Positive Formulas. A *positive* formula is an APC-formula not involving any negation or implication symbols. If the quantifiers in the formula are moved out to the front, we obtain the prenex positive formula equivalent to the original one.

Suppose $\mathfrak{P}(x_1, \ldots, x_m)$ is a positive formula of signature σ and $x_1, \ldots, x_m$ are all its free object variables. It determines an m-ary predicate $R_{\mathfrak{M}}$ on each algebraic system $\mathfrak{M}$ of signature σ called the *positive-formula predicate.*

Theorem 1. *Let $R(x_1, \ldots, x_m)$ be the positive formula predicate determined by the prenex positive formula*

$$(Q_1 x_{m+1}) \ldots (Q_n x_{m+n}) \mathfrak{A}(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \tag{1}$$

of signature σ and let φ be a homomorphism of an algebraic system $\mathfrak{M}$ of signature σ onto an algebraic system $\mathfrak{N}$. Then the truth of $R_{\mathfrak{M}}(x_1, \ldots, x_m)$ for $x_1, \ldots, x_m$ in $\mathfrak{M}$ implies the truth of $R_{\mathfrak{N}}(x_1\varphi, \ldots, x_m\varphi)$. In other words, positive-formula predicates are stable under homomorphisms of systems onto systems.

A positive formula can be built up by means of conjunction, disjunction and quantification from *elementary formulas* of the form $P(x_{i_1}, \ldots, x_{i_s})$ or $f = g$, where P is a predicate symbol and f and g are terms. Theorem 1 will therefore have been proved if we can show that such elementary formulas are stable under homomorphism and that the conjunction, disjunction or quantification of homomorphically stable formulas yields homomorphically stable formulas.

The homomorphic stability of the above elementary formulas was proved in Section 6.1. Suppose R_1 and R_2 are homomorphically stable predicates. If their disjunction

$$R_1(x_{i_1}, \ldots, x_{i_s}) \vee R_2(x_{j_1} \ldots, x_{j_t}) \tag{2}$$

is true for values $x_1^0, \ldots, x_k^0$ in the system $\mathfrak{M}$, then at least one member in the disjunction has to be true. Suppose, for example, $R_1(x_{i_1}^0, \ldots, x_{i_s}^0) = \mathsf{T}$. Then by hypothesis, $R_1(x_{i_1}^0\varphi, \ldots, x_{i_s}^0\varphi) = \mathsf{T}$ and so the disjunction (2) is also true in the system $\mathfrak{N}$ for $x_1 = x_1^0\varphi, \ldots, x_k = x_k^0\varphi$. In a similar way, it can be shown that the conjunction of homomorphically stable predicates yields a homomorphically stable predicate.

Consider now the predicate $(\forall x_1) R_1(x_1, x_2, \ldots, x_s)$ and suppose it is true for $x_2^0, \ldots, x_s^0$ in $\mathfrak{M}$. This means that for any x in $\mathfrak{M}$, the expression $R_1(x, x_2^0, \ldots, x_s^0)$ is true and so the expression $R_1(x\varphi, x_2^0\varphi, \ldots, x_s^0\varphi)$ is true in $\mathfrak{N}$ for any x in $\mathfrak{M}$. Since φ is a homomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$, $x\varphi$ varies over the whole system $\mathfrak{N}$ as x varies over $\mathfrak{M}$. In other words, $R_1(x, x_2^0\varphi, \ldots, x_s^0\varphi)$ is true for any x in $\mathfrak{N}$. Hence, the expression $(\forall x) R_1(x, x_2^0\varphi, \ldots, x_s^0\varphi)$ is true in $\mathfrak{N}$, i.e., the formula predicate $(\forall x) R_1(x, x_2, \ldots, x_s)$ is homomorphically stable.

In a similar way, the predicate $(\exists x) R x_1(x, x_2, \ldots, x_s)$ can be shown to be homomorphically stable.

Corollary 1. *If a closed positive formula is true in some algebraic system $\mathfrak{M}$, then it is true in any homomorphic image of $\mathfrak{M}$.*

A formula is said to be *negative* if it is concocted out of conjunctions, disjunctions and quantifications of expressions of the form

$$f \neq g, \quad \neg P_j(\mathfrak{x}_1, \ldots, \mathfrak{x}_m),$$

wherein the $\mathfrak{x}_i$ are object variables, f and g are terms and the P_j are predicate symbols.

Suppose $\mathfrak{F}$ is a positive formula. By applying the standard operations of negation extension and removal of quantifiers, we can reduce $\neg\mathfrak{F}$ to negative form. Conversely, if $\mathfrak{F}$ is a negative formula, then $\neg\mathfrak{F}$ can be reduced to positive form by means of such transformations.

Corollary 2. *Let $\mathfrak{F}(y_1, \ldots, y_n)$ be a negative formula of signature σ with free variables $y_1, \ldots, y_n$ and let φ be a homomorphism of an algebraic system $\mathfrak{M}$ of signature σ onto an algebraic system $\mathfrak{N}$. Then the truth of $\mathfrak{F}(y_1\varphi, \ldots, y_n\varphi)$ for any $y_1, \ldots, y_n$ in $\mathfrak{M}$ implies the truth of $\mathfrak{F}(y_1, \ldots, y_n)$ in $\mathfrak{M}$.*

In particular, *if a closed negative formula is true in $\mathfrak{N}$, then it is true in any homomorphic pre-image of $\mathfrak{N}$.*

For, if $\mathfrak{F}(y_1, \ldots, y_n)$ were false, then the positive formula $\neg\mathfrak{F}(y_1, \ldots, y_n)$ would be true. Thus $\neg\mathfrak{F}(y_1\varphi, \ldots, y_n\varphi)$ would also be true, which is a contradiction.

The converse of Theorem 1 is valid (as is that of Corollary 2) but its proof requires more subtle reasoning (see Lyndon [35]).

The question of the existence of a decision procedure for telling whether or not a formula $\mathfrak{F}$ is equivalent to a positive formula is easily shown to be answered in the negative.

Theorem 2. *There is no decision procedure for discerning whether or not an arbitrary closed formula $\mathfrak{F}$ is equivalent to a positive formula.*

By Church's theorem, there is no decision procedure for telling which APC-formulas are identically true. Let us show that if there were a decision procedure $\mathscr{A}$ permitting one to ascertain whether or not an arbitrarily given formula is equivalent to a positive formula, then $\mathscr{A}$ could be used to also ascertain the formulas that are identically true.

Thus, if a given formula $\mathfrak{F}$ were identically true, it would be equivalent to a positive formula, for example, the formula $(\forall x)\, x = x$. On the other hand, suppose $\mathfrak{F}$ is equivalent to some positive formula. Hilbert's process enables us to find effectively one of these formulas. Let $\mathfrak{F}_1$ be a positive formula equivalent to $\mathfrak{F}$. It remains to determine whether $\mathfrak{F}_1$ is identically true. This question is equivalent to: Is the negative formula $\neg \mathfrak{F}_1$ satisfiable? But if a negative formula is true in a system, it is also true in any of its homomorphic pre-images and, in particular, in an absolutely free algebraic system with an infinite number of generators (see Section 12.2). Thus, the question has been reduced to the following: To find out whether the formula $\neg \mathfrak{F}_1$ is true or false in an absolutely free system of infinite rank. A decision procedure for answering this question was indicated in [44].

7.5 Multiplicatively Stable Formulas. A *Horn* formula (see A. Horn [22]) is a prenex formula whose quantifier-free portion is the conjunction of terms each of which is either (*a*) an elementary formula, i.e., $f = g$ or an atomic formula of the form $P(x_1, \ldots, x_s)$ (here P is a signature predicate symbol and f and g are terms); (b) the disjunction of one such elementary formula and several negations of elementary formulas; or (c) the disjunction of negations of elementary formulas. For example, the formula

$$(\forall x)(\exists y)(\forall z)(f_0 = g_0 \,\&\, f_1 = g_1 \,\&\, f_2 = g_2 \vee f_3 \neq g_3 \vee f_4 \neq g_4),$$

wherein the f_i and g_i are terms, is a Horn formula.

If the object variables $x_1, \ldots, x_n$ in a Horn formula $\mathfrak{H}(x_1, \ldots, x_m)$ of signature σ are free, then it determines an m-ary predicate in every algebraic system $\mathfrak{M}$ of signature σ, which will be called a *Horn predicate.*

Suppose now that each system $\mathfrak{M}$ has a predicate $P_{\mathfrak{M}}(x_1, \ldots, x_m)$ defined in it in some way. The predicate is called *multiplicatively stable* if it satisfies the following condition.

Let $\mathfrak{M}$ be the direct product of a family of models $\mathfrak{M}_\alpha$ $(\alpha \in A)$ of signature σ and let $a_1, \ldots, a_m$ be arbitrary elements of $\mathfrak{M}$. If to each $\alpha \in A$,

$$P_{\mathfrak{M}_\alpha}(a_1\pi_\alpha, \ldots, a_m\pi_\alpha) = \mathsf{T}$$

(π_α being the projection of $\mathfrak{M}$ on $\mathfrak{M}_\alpha$), then

$$P_{\mathfrak{M}}(a_1, \ldots, a_m) = \mathsf{T}.$$

The main property of Horn predicates is expressed by the following theorem.

Theorem 1 (A. Horn [22]). *Every Horn predicate is multiplicatively stable. In particular, if a closed Horn formula $\mathfrak{F}$ is true in each factor $\mathfrak{M}_\alpha$ of a direct product $\mathfrak{M} = \prod \mathfrak{M}_\alpha$ $(\alpha \in A)$, then $\mathfrak{F}$ is true in the direct product itself.*

To prove this, we introduce an auxiliary concept. A predicate $P_{\mathfrak{M}}(y_1, \ldots, y_n)$ defined in each system $\mathfrak{M}$ of signature σ is said to be *strictly multiplicatively stable* if it is multiplicatively stable and if its truth at some point $y_1, \ldots, y_n$ in $\prod \mathfrak{M}_\alpha$ implies its truth in each factor $\mathfrak{M}_\alpha$ at the corresponding point $y_1\pi_\alpha, \ldots, y_n\pi_\alpha$.

The fact that the atomic formulas

$$P_i(y_1, \ldots, y_n), \quad y_i = F(y_1, \ldots, y_n), \tag{1}$$

where P_i and F_i are signature symbols, are strictly multiplicatively stable follows from the definition of direct product. Equally obvious is the fact that the conjunction of (strictly) multiplicatively stable predicates is a (strictly) multiplicatively stable predicate.

Let us show that quantification of (strictly) multiplicatively stable predicates yields (strictly) multiplicatively stable predicates.

For, suppose $P(y_1, \ldots, y_n)$ is a strictly multiplicatively stable predicate and suppose that $(\exists y_1)P(y_1, y_2, \ldots, y_n)$ is true in the direct product

$$\mathfrak{M} = \prod \mathfrak{M}_\alpha \quad (\alpha \in A)$$

for some $y_2, \ldots, y_n$ in $\mathfrak{M}$. This means that for suitable y in $\mathfrak{M}$, $P(y, y_2, \ldots, y_n) = \mathsf{T}$. The fact that the predicate P is strictly stable hence implies

$$P(y\pi_\alpha, y_2\pi_\alpha, \ldots, y_n\pi_\alpha) = \mathsf{T} \tag{2}$$

for each $\alpha \in A$, i.e., for each $\alpha \in A$,

$$(\exists y)P_{\mathfrak{M}_\alpha}(y, y_2\pi_\alpha, \ldots, y_n\pi_\alpha) = \mathsf{T} \tag{3}$$

Conversely, suppose (3) holds for fixed $y_2, \ldots, y_n$ in $\mathfrak{M}$ and for each $\alpha \in A$, i.e., for suitable y_α in $\mathfrak{M}_\alpha$,

$$P(y_\alpha, y_2\pi_\alpha, \ldots, y_n\pi_\alpha) = \mathsf{T}.$$

Choose an element y in $\mathfrak{M}$ such that $y\pi_\alpha = y_\alpha$ for each $\alpha \in A$. From (2) we conclude that $P(y, y_2, \ldots, y_n) = \mathsf{T}$ and so $(\exists x)P(y, y_2, \ldots, y_n) = \mathsf{T}$.

We have proved that existential quantification of a (strictly) stable predicate again yields a (strictly) stable predicate. Similar reasoning shows that universal quantification also preserves (strict) stability.

We now wish to prove that strict multiplicative stability of the predicate $P(y_1, \ldots, y_n)$ and multiplicative stability of $Q(y_1, \ldots, y_n)$ imply the multiplicative stability of the implication

$$P(y_1, \ldots, y_n) \to Q(y_1, \ldots, y_n).$$

Suppose that $P(y_1, \ldots, y_n) = \mathsf{T}$ for $y_1, \ldots, y_n$ in the direct product $\mathfrak{M}$ and that the implication

$$P(y_1\pi_\alpha, \ldots, y_n\pi_\alpha) \to Q(y_1\pi_\alpha, \ldots, y_n\pi_\alpha)$$

is true for each $\alpha \in A$. Since P is strictly stable, it follows that $P(y_1\pi_\alpha, \ldots, y_n\pi_\alpha) = \mathsf{T}$ and so $Q(y_1\pi_\alpha, \ldots, y_n\pi_\alpha) = \mathsf{T}$. From the fact that Q is stable, we now can conclude that

$$Q(y_1, \ldots, y_n) = \mathsf{T}.$$

Finally, we point out that the negation of a strictly multiplicatively stable predicate yields a multiplicatively stable predicate.

Indeed, $\neg P$ is equivalent to the implication $P \to x \neq x$, which is multiplicatively stable by the previous result.

Theorem 1 is now an immediate consequence of the statements proved above. For, by Section 7.1, a formula of the form

$$f(y_1, \ldots, y_n) = g(y_1, \ldots, y_n), \tag{4}$$

in which f and g are terms in $y_1, \ldots, y_n$, is equivalent to an $\exists$-formula whose quantifier-free portion is the conjunction of elementary expressions of the form (1). These elementary expressions are strictly multiplicatively stable. Also strictly stable will be their conjunction and the quantified formula (4). Thus, formulas such as (4) are strictly multiplicatively stable. A Horn formula is the quantification of the conjunction of terms each being a formula of the form (1) or (4), or the negation of such a formula or their conjunction, or a formula of the form

$$A_1 \,\&\, A_2 \,\&\ldots\&\, A_p \to A_{p+1},$$

where the A_i are formulas of the form (1) or (4). Since formulas of the form (1) or (4) are strictly multiplicatively stable, each term in a Horn formula is multiplicatively stable. Therefore the conjunction of the specified terms and its quantification are multiplicatively stable, as required.

The converse of Horn's theorem is false. Namely, Chang and Morel [10] have shown that there exist multiplicatively stable $\forall\exists$-formulas equivalent to no Horn formula (see Section 8.4). As for $\forall$-formulas and $\exists$-formulas, the converse of Horn's theorem is valid for them. We shall confine ourselve here to the consideration of $\forall$-formulas, which are more important in applications.

Theorem 2. *Let $\mathfrak{K}$ be a class of algebraic systems possessing the property that the direct product of any two systems in $\mathfrak{K}$ belongs to $\mathfrak{K}$. Then each $\forall$-formula $\mathfrak{F}(y_1, \ldots, y_n)$ multiplicatively stable in $\mathfrak{K}$ is equivalent in $\mathfrak{K}$ to a universal Horn formula.*

We start by representing $\mathfrak{F}$ in the form

$$(\forall x_1, \ldots, x_m)(U_1 \& \ldots \& U_r), \tag{5}$$

where U_λ is the formula

$$\neg A_{\lambda 1} \vee \ldots \vee \neg A_{\lambda p} \vee A_{\lambda, p+1} \vee \ldots \vee A_{\lambda q} \tag{6}$$

and the $A_{\lambda j}$ are elementary formulas such as (1) or (4). Assume that the number $q - p$ in (6) exceeds 1. Let $\mathfrak{F}_k$ be the formula obtained from $\mathfrak{F}$ by deleting from its member (6) the k-th expression of the form $A_{\lambda, p+k}$. The formula

$$\mathfrak{F}_k \to \mathfrak{F}$$

is clearly identically true. Let us show that the formula

$$\mathfrak{F} \to \mathfrak{F}_k \tag{7}$$

is also true in $\mathfrak{K}$ for some k.

Suppose to the contrary that for each k, formula (7) is false in class $\mathfrak{K}$. This means that $\mathfrak{K}$ has a system $\mathfrak{M}_k$ containing elements $y_1^{(k)}, \ldots, y_n^{(k)}$ such that $\mathfrak{F}$ is true and $\mathfrak{F}_k$ is false in $\mathfrak{M}_k$ at the point $y_1^{(k)}, \ldots, y_n^{(k)}$. The fact that $\mathfrak{F}$ is true implies that it is true in all its subformulas $(\forall x_1 \ldots x_m) U_i$. Since all of these subformulas except the λ-th one belong to the make-up of $\mathfrak{F}_k$, the falsity of $\mathfrak{F}_k$ implies that the subformula (6) with the deleted member is false. Hence $\mathfrak{M}_k$ has elements $x_1^{(k)}, \ldots, x_m^{(k)}$ such that the expressions $A_{\lambda 1}, \ldots, A_{\lambda p}$ are all true at the point $y_1^{(k)}, \ldots, y_n^{(k)}, x_1^{(k)}, \ldots, x_m^{(k)}$ but the expressions $A_{\lambda, p+\alpha}$ $(\alpha \neq k)$ are false.

Consider the direct product $\mathfrak{M}$ of the systems $\mathfrak{M}_k$ $(k = 1, \ldots, q - p)$. Denote by y_i the sequence $(y_i^{(1)}, \ldots, y_i^{(q-p)})$ $(i = 1, \ldots, n)$ and by x_j the sequence $(x_j^{(1)}, \ldots, x_j^{(q-p)})$ $(j = 1, \ldots, m)$. The sequences $y_1, \ldots, y_n$, $x_1, \ldots, x_m$ are elements of $\mathfrak{M}$. The expressions $A_{\lambda 1}, \ldots, A_{\lambda q}$ were true at the points $y_1^{(k)}, \ldots, y_n^{(k)}, x_1^{(k)}, \ldots, x_m^{(k)}$ $(k = 1, \ldots, q - p)$ and they are therefore also true at the point $\mathfrak{a} = (y_1, \ldots, y_n, x_1, \ldots, x_m)$. On the other hand, each of the expressions $A_{\lambda, p+1}, \ldots, A_{\lambda q}$ is false at a suitable projection $\mathfrak{a}\pi_k = (y_1^{(k)}, \ldots, y_n^{(k)}, x_1^{(k)}, \ldots, x_m^{(k)})$ of point $\mathfrak{a}$. Therefore $A_{\lambda, p+1}, \ldots, A_{\lambda q}$ are all false at point $\mathfrak{a}$. We see that U_λ is false at point $\mathfrak{a}$. Hence, $\mathfrak{F}$ is false at point $(y_1, \ldots, y_n)$. But this contradicts the multiplicative stability of $\mathfrak{F}$ since $\mathfrak{F}(y_1\pi_k, \ldots, y_n\pi_k)$ is a true formula for each $k = 1, \ldots, q - p$.

Thus if a multiplicatively stable $\forall$-formula $\mathfrak{F}$ is not of Horn form, then it is equivalent in class $\mathfrak{K}$ to a shorter $\forall$-formula $\mathfrak{F}_k$ multiplicatively stable in $\mathfrak{K}$. If $\mathfrak{F}_k$ is not in Horn form, then on carrying out the above deletion process, we obtain a still shorter formula $\mathfrak{F}_{kr}$ and so on. After a finite number of steps, the process must terminate and the result is a formula satisfying the requirements of Theorem 2.

In this section and the preceding ones, we defined several kinds of stability: stability under passage to subsystems, homomorphic stability, and multiplicative stability. Of the formulas possessing these kinds of stability, the so-called identical relations or briefly, the identities and conditional identities (quasi-identities) are of particular importance.

By definition, an *identity* is a formula of the form

$$(\forall x_1 \ldots x_n)(f = g) \quad \text{or} \quad (\forall x_1 \ldots x_n) P(\mathfrak{a}_1, \ldots, \mathfrak{a}_m), \tag{8}$$

wherein f, g, $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are terms in $x_1, \ldots, x_n$ and P is a signature predicate symbol.

A *conditional identity (quasi-identity)* is a formula of the form

$$(\forall x_1 \ldots x_n)(A_1 \& \cdots \& A_p \to A),$$

wherein $A_1, \ldots, A_p$ and A stand for elementary formulas of the form

$$f = g \quad \text{or} \quad P(\mathfrak{a}_1, \ldots, \mathfrak{a}_m).$$

From the results of this section, it is apparent that identical relations are stable under passage to subsystems and that they are homomorphically and multiplicatively stable. Conditional identities are stable under passage to subsystems and are multiplicatively stable. But in general, they do not possess homomorphic stability.

We further point out that each identity in (8) is equivalent to the conditional identity

$$(\forall x_1 \ldots x_n)(x_1 = x_1 \to f = g)$$

or the conditional identity

$$(\forall x_1 \ldots x_n)\big(x_1 = x_1 \to P(a_1, \ldots, a_m)\big).$$

Therefore identities may be viewed as special forms of conditional identities.

Several problems have been posed in the literature on the existence of decision procedures related in a natural way to the concepts of Horn formula and multiplicative stability. For example:

(a) Does there exist a decision procedure permitting one to tell whether an arbitrary closed formula possesses the property of multiplicative stability?

(b) Does there exist a decision procedure permitting one to tell whether an arbitrary closed formula is equivalent to a suitable Horn formula?

The questions (a) and (b) may be specialized to considering not any closed formulas but just the formulas of fixed signature. Questions (a) and (b) may also be modified differently. For example, let us agree to call a closed formula $\mathfrak{F}$ *square stable* if the truth of $\mathfrak{F}$ in an arbitrary system $\mathfrak{M}$ implies its truth in the direct product $\mathfrak{M} \times \mathfrak{M}$. It is natural to tack on the following question to (a) and (b):

(c) Does there exist a decision procedure permitting one to tell whether an arbitrary closed formula is square stable?

Questions (b) and (c) and certain modifications of them were solved in the negative by Feferman (for formulas of any signature) and by Machover [37] (for formulas of signature containing either one k-ary predicate symbol, $k \geqq 3$, or two predicate symbols, one binary and the other not nullary). We shall confine our considerations to a proof of the following theorem.

Theorem 3 (Almagambetov [4]). *No decision procedure $\mathscr{A}$ exists for telling whether a formula in the class of closed APC-formulas whose signature consists of just one binary predicate symbol and which involve no equality signs is square stable.*

Let $\mathfrak{K}$ be the class of models whose signature consists of one binary predicate symbol P and in which the formula

$$\mathfrak{A} \overset{df}{=} (\forall xy)\, P(x, y)$$

is true.

Let $\mathfrak{L}$ be the class of models with signature $\langle P \rangle$ in which the formula

$$\mathfrak{B} \stackrel{df}{=} (\forall xy)\big(P(x, y) \vee P(y, x)\big)$$

is true.

Consider any closed formula

$$\mathfrak{F} \stackrel{df}{=} (Q_1 x_1) \ldots (Q_m x_m)\, \mathfrak{R}(x_1, \ldots, x_m) \tag{9}$$

of signature $\langle P \rangle$ not containing equality signs. To find out if this formula is true or false in some model $\mathfrak{M}$ belonging to $\mathfrak{K}$, we merely have to replace all expressions of the form $P(x_i, x_j)$ in the quantifier-free portion of $\mathfrak{R}$ by their value T and then with the help of the truth table in Section 6.2 to determine the value of the resulting formula. If the value is T, then $\mathfrak{F}$ is not only true in model $\mathfrak{M}$ but also in any model of class $\mathfrak{K}$. If the value of $\mathfrak{R}$ is F, then $\mathfrak{F}$ is not only false in $\mathfrak{M}$ but also in any model of $\mathfrak{K}$. Consequently, we have obtained a decision procedure for telling whether or not each formula of the form (9) is true or false in all the models belonging to $\mathfrak{K}$.

In contradiction to this, it is known that there is no decision procedure by means of which one can tell whether each formula of the form (9) is or is not satisfiable in all models belonging to $\mathfrak{L}$.

Therefore to prove Theorem 3, we merely have to show that if a decision procedure such as $\mathscr{A}$ really existed, then by using it we would be able to decide whether or not each formula of the form (9) is satisfiable in the models belonging to $\mathfrak{L}$.

Thus, suppose $\mathfrak{F}$ is some given formula of the form (9). We first find out whether $\mathfrak{F}$ is true in all the models belonging to $\mathfrak{K}$. If yes, then $\mathfrak{F}$ is also satisfiable in $\mathfrak{L}$ (since $\mathfrak{K} \subset \mathfrak{L}$) and the question of the satisfiability of $\mathfrak{F}$ in $\mathfrak{L}$ is decided.

Suppose $\mathfrak{F}$ is false in $\mathfrak{K}$. Using $\mathscr{A}$, we can decide whether the formula $\mathfrak{F} \,\&\, \mathfrak{B}$ is square stable.

Suppose it is not. Then $\mathfrak{F}$ is satisfiable in $\mathfrak{L}$. Indeed, the fact that the formula $\mathfrak{F} \,\&\, \mathfrak{B}$ is not square stable means there is a model $\mathfrak{M}$ such that $\mathfrak{F} \,\&\, \mathfrak{B}$ is true in it but false in $\mathfrak{M}^2$. From the truth of $\mathfrak{F} \,\&\, \mathfrak{B}$ in $\mathfrak{M}$ it follows that $\mathfrak{F}$ and $\mathfrak{B}$ are individually true in $\mathfrak{M}$, i.e., $\mathfrak{F}$ is true in model $\mathfrak{M}$ of class $\mathfrak{L}$.

Suppose, finally, that formula $\mathfrak{F} \,\&\, \mathfrak{B}$ is square stable. We wish to show that $\mathfrak{F}$ is then not satisfiable in $\mathfrak{L}$. Suppose to the contrary that $\mathfrak{F}$ is true in some model $\mathfrak{M}$ belonging to $\mathfrak{L}$. By hypothesis, $\mathfrak{F}$ is false in all $\mathfrak{K}$-models and so $\mathfrak{M} \notin \mathfrak{K}$, i.e., $\mathfrak{M}$ has elements a and b for which $P(a, b) = \mathsf{F}$. Consider the elements $\mathfrak{a} = (a, b)$ and $\mathfrak{b} = (b, a)$ in model $\mathfrak{M}^2$. Since $P(a, b) = \mathsf{F}$, we have

$$P(\mathfrak{a}, \mathfrak{b}) = P(\mathfrak{b}, \mathfrak{a}) = \mathsf{F}$$

and hence, $\mathfrak{B}$ is false in $\mathfrak{M}^2$. But in view of the square stability of $\mathfrak{F} \& \mathfrak{B}$, the formula $\mathfrak{F} \& \mathfrak{B}$ must also be true in $\mathfrak{M}^2$. The resulting contradiction shows that the statement about the unsatisfiability of $\mathfrak{F}$ in $\mathfrak{L}$ is true.

Perfectly similar reasoning leads to the following result.

Theorem 4 (Almagambetov [4]). *No decision procedure exists for telling whether an arbitrary closed formula whose signature consists of one binary predicate symbol and which involves no equality signs is multiplicatively stable.*

Chapter IV

Products and Complete Classes

8. Filters and Filtered Products

The prominent place held by the direct product in group theory, ring theory and in the theories of other classical algebraic objects, depends largely on the fact that the direct product of groups is a group, the direct product of rings is a ring, etc. For the theory of first-order languages, the concept is less convenient since the direct product of systems having a certain first-order property may in general not possess that property. At the end of the 1950's, so-called ultraproducts of systems were introduced. They in full measure satisfied the requirement that first-order properties be preserved. At the present time the technique of the ultraproduct construction is playing a great role in the solution of rather delicate problems of first-order theory. The basic ideas and results of the theory of ultraproducts are elucidated in this section.

8.1 Filters and Ultrafilters. A *filter* over a nonempty set I is any collection D of subsets of I satisfying the conditions:

i) the intersection of any two subsets belonging to D is a member of D;

ii) all supersets of a given subset in D also belong to D;

iii) the empty subset $\varnothing$ does not belong to D.

From conditions i), ii) it immediately follows that the intersection of any finite number of sets belonging to a filter also belongs to that filter and that the base set I belongs to every filter over I. A filter D over I will be denoted either by the single letter D or by the pair $\langle I, D\rangle$.

The collection of all supersets of a fixed nonempty set $M \subseteq I$ obviously satisfies conditions i), ii), iii) and is therefore a filter. In particular, the set I taken by itself is a filter. Filters of this kind are called *principal*. All other filters are called *nonprincipal*. Clearly, *a filter D is principal if and only if it contains the intersection of all its sets.*

Since a finite set has only a finite number of subsets, and the intersection of a finite number of subsets of a filter belongs to the filter, *every filter over a finite set I is principal.*

There are nonprincipal filters over each infinite set I. The simplest of these are called *Fréchet filters* and are constructed in the following way. Let $\mathfrak{n}$ be any infinite cardinal not exceeding $|I|$. Denote by $D_{\mathfrak{n}}$ the collection of all subsets of I each having a complement of cardinality strictly less than $\mathfrak{n}$. From the results of Section 2.6, it immediately follows that the collection $D_{\mathfrak{n}}$ satisfies conditions i), ii), iii), and hence $D_{\mathfrak{n}}$ is a filter over I. Sets of the form $I - \{\alpha\}$ $(\alpha \in I)$ belong to $D_{\mathfrak{n}}$. The intersection of all these sets and consequently the intersection of all sets in $D_{\mathfrak{n}}$ is empty. Therefore $D_{\mathfrak{n}}$ is a nonprincipal filter.

For example, over the set of natural numbers, the only Fréchet filter is the collection of all subsets with finite complement. Over the set R of all real numbers, Fréchet filters are formed by a) the collection of all cofinite subsets, i.e., all subsets having a finite complement and b) the collection of all subsets whose complement is finite or countable. The first of these filters is a subset of the second. If one accepts the continuum hypothesis (Section 2.6), then there are no other Fréchet filters over R.

An example of a nonprincipal filter which is not a Fréchet filter is the collection of sets of natural numbers whose complement contains only a finite number of even numbers. Another is the collection of sets of rational numbers each consisting of all rational numbers of a suitable interval containing a fixed irrational number θ. The latter filter is viewed as a filter over the set of rational numbers.

In accordance with Section 2.6, 2^I means the set of all subsets of I. Filters are subsets of the set 2^I which satisfy the special conditions i)-iii). The family of all filters over I is partially ordered by inclusion. The following theorems express the simplest properties of this family.

Theorem 1. *The intersection of any nonempty family of filters over I is a filter over I. The union of a chain of filters over I (ordered by inclusion) is a filter over I.*

The proofs are evident.

Theorem 2. *A collection Σ of subsets of a set I is part of a filter over I if and only if the intersection of any finite number of subsets in Σ is not empty.*

Necessity is obvious. We shall prove sufficiency. Suppose the intersection of any finite number of sets in Σ is nonempty. Denote by D the collection of all those subsets of I each of which contains the intersection of a finite number of sets of Σ. Since these intersections are assumed nonempty, D does not contain $\varnothing$. Next, if a subset $A \subseteq I$ contains the intersection $A_1 \cap \cdots \cap A_m$ of sets from Σ, and a subset $B \subseteq I$ contains the intersection $B_1 \cap \cdots \cap B_n$ of sets from Σ, then $A \cap B$ contains the intersection $A_1 \cap \cdots \cap A_m \cap B_1 \cap \cdots \cap B_n$ and therefore belongs

to the collection D. Thus D satisfies conditions i) and iii). Clearly, D also satisfies ii) and hence D is the desired filter containing Σ.

Maximal filters, i.e. not contained in any other filter over I, are called *ultrafilters* over I.

The principal ultrafilters over I are clearly the collections of all subsets that contain some fixed element a of I. The existence of nonprincipal ultrafilters over infinite sets follows from the next proposition.

Theorem 3. *A collection Σ of subsets of the set I is contained in an ultrafilter over I if and only if the intersection of any finite number of subsets in Σ is nonempty. In particular, each filter over I is contained in an ultrafilter over I.*

The proof need only establish the sufficiency of the conditions. Suppose that the intersection of any finite number of sets in Σ is nonempty. By Theorem 2, Σ is contained in some filter D over I. Since the union of a chain of filters is a filter, by Zorn's lemma (Section 1.5) every filter, including D, is contained in some maximal filter, i.e., in an ultrafilter.

If the intersection of all the sets in a given filter D is empty, for example, if D is a Fréchet filter, then any ultrafilter containing D will also be nonprincipal. Therefore, there are nonprincipal ultrafilters over every infinite set.

The above proof of the existence of nonprincipal ultrafilters depends upon Zorn's Lemma, i.e., on the Axiom of Choice. Without the Axiom of Choice, no nonprincipal ultrafilter (even over the set of natural numbers) has yet been constructed. It is easily shown that the extendability of a filter to an ultrafilter is equivalent to the Axiom of Choice.

A convenient test for the maximality of a filter is given in the next proposition.

Theorem 4. *A filter D over I is an ultrafilter if and only if each subset A of I either belongs to D or its complement belongs to D.*

Indeed, suppose D has the given property and suppose to the contrary that D is contained in some larger filter D_1. Let $A \in D_1$ and $A \notin D$. By hypothesis, if $A \notin D$, then $A' \in D$, where A' denotes the complement of A.

Thus, $A \in D_1$, $A' \in D_1$ and therefore $A \cap A' = \varnothing \in D_1$, which contradicts condition iii).

Conversely, suppose that D is an ultrafilter over I and

$$A \notin D, \quad A' \notin D \quad (A \subseteq I).$$

If $A \cap X = \varnothing$ for some $X \in D$, then $X \subseteq A'$ and so $A' \in D$. Therefore, $A \cap X \neq \varnothing$ for any $X \in D$ and, hence, $A \cap X_1 \cap \cdots \cap X_m \neq \varnothing$

for arbitrary $X_1, \ldots, X_m$ in D. We see that the collection Σ consisting of the set A and all the sets of the filter D satisfies the conditions of Theorem 2 and is therefore contained in some filter D_1. But D is an ultrafilter, i.e. it can not be contained in any larger filter D_1, as required.

8.2 Ultraproducts. Suppose that with each element α of set I we associate an algebraic system $\mathfrak{A}_\alpha = \langle A_\alpha, \Omega \rangle$ of fixed signature Ω. The elements of the cartesian product

$$M = \prod A_\alpha \quad (\alpha \in I)$$

of the carriers A_α of the systems are the functions f defined on I whose values satisfy $f(\alpha) \in A_\alpha$. Instead of $f(\alpha)$ we shall write f^α and call f^α the projection of f on the factor A_α.

Suppose that D is a filter over I. Introduce a binary relation $\equiv_D$ in M defined by

$$f \equiv_D g \quad \Leftrightarrow \quad \{\alpha \in I \mid f^\alpha = g^\alpha\} \in D \qquad (f, g \in M). \tag{1}$$

If $f \equiv_D g$, then we shall say that f is equivalent to g modulo the filter D.

From definition (1), it is clear that $\equiv_D$ is a symmetric relation. Since $f \equiv_D f$, $\equiv_D$ is reflexive. Let us show that it is also transitive. Suppose $f \equiv_D g$ and $g \equiv_D h$. Thus, besides the right-hand side of (1), we also have

$$\{\alpha \in I \mid g^\alpha = h^\alpha\} \in D.$$

Since

$$\{\alpha \mid f^\alpha = h^\alpha\} \supseteq \{\alpha \mid f^\alpha = g^\alpha\} \cap \{\alpha \mid g^\alpha = h^\alpha\} \in D,$$

it follows that

$$f \equiv_D h.$$

Thus, $\equiv_D$ is an equivalence relation on the set M and so we may form the quotient set $A = M/\equiv_D$. It is called the *reduced* or *filtered product* of the sets A_α (modulo D) and is denoted by $\prod A_\alpha/D$ $(\alpha \in I)$. The symbol fD $(f \in M)$ denotes the class of elements of M equivalent to f modulo D.

We now wish to define on the set $A = M/\equiv_D$ an algebraic system of signature Ω. Suppose that R is some m-ary predicate symbol in Ω. By definition we say

$$R(f_1 D, \ldots, f_m D) = \mathsf{T} \Leftrightarrow \{\alpha \mid R(f_1^\alpha, \ldots, f_m^\alpha) = \mathsf{T}\} \in D. \tag{2}$$

The truth value of $R(f_1 D, \ldots, f_m D)$ as defined by (2) does not depend on the choice of representatives $f_1, \ldots, f_m$ of the classes $f_1 D, \ldots, f_m D$. Indeed, if

$$g_i \in f_i D \quad (i = 1, \ldots, m), \; R(f_1 D, \ldots, f_m D) = \mathsf{T},$$

then

$$I_i \overset{df}{=} \{\alpha \mid g_i^\alpha = f_i^\alpha\} \in D.$$

But

$$\{\alpha \mid R(g_1^\alpha, \ldots, g_m^\alpha) = \mathsf{T}\} \supseteq I_1 \cap \cdots \cap I_m \cap \{\alpha \mid R(f_1^\alpha, \ldots, f_m^\alpha) = \mathsf{T}\} \in D,$$

and therefore the set $\{\alpha \mid R(g_1^\alpha, \ldots, g_m^\alpha) = \mathsf{T}\}$ belongs to the filter D and hence, $R(g_1 D, \ldots, g_m D) = \mathsf{T}$.

If F is an n-ary operator symbol in Ω, then in accordance with (2) we set

$$F(f_1 D, \ldots, f_n D) = fD \Leftrightarrow \{\alpha \mid F(f_1^\alpha, \ldots, f_n^\alpha) = f^\alpha\} \in D. \tag{3}$$

As in the case of the predicate symbol R, we easily see that the relation (3) determines an everywhere defined and singlevalued function F on $\prod A_\alpha/D$.

Definitions (2) and (3) convert the filtered product $A = \prod A_\alpha/D$ into an algebraic system $\langle A, \Omega\rangle$ called the *reduced* or *filtered product* of the sequence of systems $\mathfrak{A}_\alpha$ $(\alpha \in I)$ (modulo D) and denoted by any of the following:

$$\prod_{\alpha \in I} \mathfrak{A}_\alpha/D, \quad \prod \mathfrak{A}_\alpha/D \ (\alpha \in I), \quad \prod \mathfrak{A}_\alpha/D.$$

The $\mathfrak{A}_\alpha$ are called *factors* of the filtered product. If all the $\mathfrak{A}_\alpha$ are the same system $\mathfrak{A}$, then the filtered product $\prod \mathfrak{A}_\alpha/D$ is called the *I-th power of $\mathfrak{A}$ filtered modulo D* and is denoted by $\mathfrak{A}^I/D$.

If the filter $\langle I, D\rangle$ consists of only the set I, then the filtered product $\prod \mathfrak{A}_\alpha/D$ $(\alpha \in I)$ coincides with the direct product $\prod \mathfrak{A}_\alpha$ $(\alpha \in I)$ of systems $\mathfrak{A}_\alpha$ $(\alpha \in I)$ which was defined in Section 2.5. More detailed information on the behavior of filtered products under change of filters or factors is given by the following basic theorems.

Theorem 1. *If D and D_1 are filters over the same set I and $D \subseteq D_1$ then the canonical mapping*

$$i\colon fD \to fD_1 \quad (f \in \prod \mathfrak{A}_\alpha) \tag{4}$$

is a homomorphism of $\prod \mathfrak{A}_\alpha/D$ onto $\prod \mathfrak{A}_\alpha/D_1$.

From $D \subseteq D_1$ it follows that

$$f \equiv_D g \Rightarrow f \equiv_{D_1} g \quad (f, g \in \prod \mathfrak{A}_\alpha)$$

and therefore (4) is a singlevalued mapping of $\prod \mathfrak{A}_\alpha/D$ onto $\prod \mathfrak{A}_\alpha/D_1$. Moreover $D \subseteq D_1$ implies that for any $R \in \Omega$

$$R(f_1 D, \ldots, f_m D) = \mathsf{T} \Rightarrow R(f_1 D_1, \ldots, f_m D_1) = \mathsf{T}.$$

Therefore the mapping (4) is a homomorphism.

Taking for D a filter consisting only of the set I, we obtain the canonical mapping $f \to fD_1$ which is a homomorphism of the direct product $\prod \mathfrak{A}_\alpha$ onto the filtered product $\prod \mathfrak{A}_\alpha/D_1$. However, in the general case, the homomorphism (4) is not strong (Section 2.4) and hence we can not assert that the filtered product is a quotient system of the direct product modulo some congruence θ. Nevertheless, if the factors $\mathfrak{A}_\alpha$ are algebras, then the filtered product $\prod \mathfrak{A}_\alpha/D_1$ will also be an algebra. Any homomorphism of an algebra onto an algebra is a strong homomorphism. Therefore, the algebra product $\prod \mathfrak{A}_\alpha/D_1$ filtered by an arbitrary filter D_1 is a quotient algebra of the direct product $\prod \mathfrak{A}_\alpha$ modulo the congruence $\equiv_{D1}$.

Theorem 2. *Suppose that $\prod \mathfrak{A}_\alpha/D$ is a filtered product of algebraic systems $\mathfrak{A}_\alpha$ ($\alpha \in I$) and $J \in D$.* Then*)

$$\prod_{\alpha \in I} \mathfrak{A}_\alpha/D \cong \prod_{j \in J} \mathfrak{A}_j/D_J, \tag{5}$$

where D_J is the filter formed by the intersections of J with the sets of the filter D.

In particular, if D is the principal filter consisting of all supersets of $J \subseteq I$, then $\prod \mathfrak{A}_\alpha/D \cong \prod \mathfrak{A}_j$ ($j \in I$).

Consider the mapping

$$f \to f_J \quad (f \in \prod \mathfrak{A}_\alpha),$$

where f_J is the restriction of f to the set J. From definition (1) and the construction of the filter D_J, it is clear that

$$f \equiv_D g \Leftrightarrow f_J \equiv_{D_J} g_J \quad (f, g \in \prod \mathfrak{A}_\alpha)$$

and for each $R \in \Omega$,

$$R(f_1 D, \ldots, f_n D) = \mathsf{T} \Leftrightarrow R\big((f_1)_J D_J, \ldots, (f_n)_J D_J\big) = \mathsf{T}.$$

Therefore the mapping $fD \to f_J D_J$ is the required isomorphism (5).

Theorem 3. *If the intersection J of all the sets of the filter $\langle I, D\rangle$ is nonempty and does not belong to D, then*

$$\prod \mathfrak{A}_\alpha/D \cong \prod_{j \in J} \mathfrak{A}_j \cdot \prod_{\beta \in J'} \mathfrak{A}_\beta/D_{J'},$$

where $D_{J'}$ is the filter over $J' = I - J$ formed by the intersections of J' with all the sets of the filter D.

The proof is analogous to that of Theorem 2.

---

*) If $\mathfrak{A}$ and $\mathfrak{B}$ are algebraic systems, then $\mathfrak{A} \cong \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.

From the assumptions in Theorem 3, we see that the intersection of all the sets of the filter D_J is empty and hence, Theorem 3 reduces the study of arbitrary filtered products to the study of direct products and filtered products modulo a filter all of whose sets have an empty intersection.

Theorem 4. *Suppose we are given sequences* $\mathfrak{A}_\alpha (\alpha \in I)$ *and* $\mathfrak{B}_\alpha (\alpha \in I)$ *of algebraic systems and homomorphisms*

$$\varphi_\alpha \colon \mathfrak{A}_\alpha \to \mathfrak{B}_\alpha$$

of the systems $\mathfrak{A}_\alpha$ *into the systems* $\mathfrak{B}_\alpha$. *Then the mapping* $\varphi \colon fD \to (f\varphi)D$, *where* $f\varphi$ *is defined by*

$$(f\varphi)(\alpha) = f(\alpha)\varphi_\alpha \qquad (f \in \prod \mathfrak{A}_\alpha),$$

is a homomorphism of $\prod \mathfrak{A}_\alpha/D$ *into* $\prod \mathfrak{B}_\alpha/D$ *for any filter* $\langle I, D\rangle$. *If the* φ_α *are strong homomorphisms, isomorphisms, or onto, then so is the mapping* φ.

The proof of this theorem follows directly from the definitions.

Theorem 5. *Suppose we have a filtered product* $\prod \mathfrak{A}_\alpha/D (\alpha \in I)$ and *one-to-one mapping* $\sigma \colon I \to J$ *of the set* I *onto some set* J. *Denote by* D^σ *the filter over* J *formed by the* σ*-images of the sets of the filter* D. *Then*

$$\prod_{\alpha \in I} \mathfrak{A}_\alpha/D \cong \prod_{\beta \in J} \mathfrak{A}_{\beta\sigma^{-1}}/D^\sigma. \tag{6}$$

It is easy to check that the mapping $f \to f^\varphi$, where

$$f^\varphi(\beta) = f(\beta\sigma^{-1}) \qquad (f \in \prod \mathfrak{A}_\alpha),$$

is precisely the desired isomorphism (6).

Theorems 4 and 5 show that up to isomorphism a filtered product does not depend upon what way the systems are numbered but is determined by the abstract properties of the factors and the filter.

We now ask the following fundamental question. Consider a filtered product $\prod \mathfrak{A}_\alpha/D$ of algebraic systems $\mathfrak{A}_\alpha$ with signature Ω. Suppose that a fixed closed first-order formula $\mathfrak{F}$ of signature Ω is true in each $\mathfrak{A}_\alpha$. Under what conditions on the formula $\mathfrak{F}$ or on the filter D can one assert that $\mathfrak{F}$ is also true in the filtered product? In order to answer this question, we introduce two auxiliary notions.

Assume as before that we are given a filter $\langle I, D\rangle$ and a mapping $\alpha \to \mathfrak{A}_\alpha = \langle A_\alpha, \Omega\rangle$ of the set I into the class of algebraic systems of fixed signature Ω. Set

$$M = \prod A_\alpha, \qquad \mathfrak{A} = \prod \mathfrak{A}_\alpha/D \qquad (\alpha \in I).$$

Consider an n-ary predicate symbol R (possibly nullary) not necessarily contained in Ω. For each $\alpha \in I$, let the value of R in $\mathfrak{A}_\alpha$ be an n-ary predicate $R^\alpha(x_1, \ldots, x_n)$ defined in $\mathfrak{A}_\alpha$. Also, suppose there is given a certain n-ary predicate R^* in $\mathfrak{A}$ which will be called the *value* of R in $\mathfrak{A}$.

The predicate R (i.e. the collection of all its values in the $\mathfrak{A}_\alpha$ and in $\mathfrak{A}$) is said to be *conditionally filtered* (in D) if for any $f_1, \ldots, f_n$ in M

$$\{\alpha \mid R^\alpha(f_1^\alpha, \ldots, f_n^\alpha) = \mathsf{T}\} \in D \Rightarrow R^*(f_1 D, \ldots, f_n D) = \mathsf{T}. \tag{7}$$

The predicate R will be called *filtered* (in D) if

$$\{\alpha \mid R^\alpha(f_1^\alpha, \ldots, f_n^\alpha) = \mathsf{T}\} \in D \Leftrightarrow R^*(f_1 D, \ldots, f_n D) = \mathsf{T}. \tag{8}$$

An operation F defined in $\mathfrak{A}_\alpha$ and $\mathfrak{A}$ will be called filtered or conditionally filtered if its corresponding predicate is filtered or conditionally filtered.

Comparing formula (8) with (2) and (3) we see that all the predicates and operations of the signature are necessarily filtered in any filter $\langle I, D\rangle$.

Consider now an arbitrary first-order formula $\mathfrak{P}(R_1, \ldots, R_s; x_1, \ldots, x_n)$ whose free object variables lie among the variables $x_1, \ldots, x_n$, and whose predicate (and operator) variables are $R_1, \ldots, R_s$. Suppose that the values of $R_1, \ldots, R_s$ in the systems $\mathfrak{A}$ and $\mathfrak{A}_\alpha (\alpha \in I)$ have already been defined. Then the above formula determines in each system $\mathfrak{A}$, $\mathfrak{A}_\alpha$ $(\alpha \in I)$ an n-ary predicate

$$R(x_1, \ldots, x_n) \overset{df}{=} \mathfrak{P}(R_1, \ldots, R_s; x_1, \ldots, x_n),$$

which may or may not be filtered in D depending on the form of the formula $\mathfrak{P}$ and the nature of the initially given predicates $R_1, \ldots, R_s$, which may in turn also be given by certain formulas.

The following lemmas establish this dependence in several simple cases.

Lemma 6. *If $\mathscr{A}(x_1, \ldots, x_n)$ and $\mathscr{B}(x_1, \ldots, x_n)$ are formulas defining conditionally filtered predicates (for some filter D), then the formulas*

$$(\forall x_1)\mathscr{A}, \quad (\exists x_1)\mathscr{A}, \quad \mathscr{A} \,\&\, \mathscr{B}$$

define predicates which are also conditionally filtered in D.

We choose fixed values for $x_2, \ldots, x_n$ in M which we shall not write explicitly in formulas below. Suppose

$$\{\alpha \mid (\forall x_1)\mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D. \tag{9}$$

Choose an arbitrary value a for x_1 in M. From (9) it follows that

$$\{\alpha \mid (\forall x_1)\mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \subseteq \{\alpha \mid \mathscr{A}(a^\alpha) \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D.$$

Since the predicate $\mathscr{A}$ is conditionally filtered, (9) implies that $\mathscr{A}(aD)$ is true in A and therefore the formula $(\forall x_1)\mathscr{A}$ is true in A.

In a similar way, one proves that the formulas $(\exists x_1)\mathscr{A}$ and $\mathscr{A} \,\&\, \mathscr{B}$ are conditionally filtered.

Lemma 7. *If $\mathscr{A}(x_1, \ldots, x_n)$ and $\mathscr{B}(x_1, \ldots, x_n)$ are formulas defining filtered predicates, then the formulas $\mathscr{A} \,\&\, \mathscr{B}$ and $(\exists x_1)\mathscr{A}$ also define filtered predicates.*

The indicated predicates are already known to be conditionally filtered. Also, we have

$$\mathscr{A} \,\&\, \mathscr{B} \underset{\mathfrak{A}}{=} \mathsf{T} \Rightarrow \left(\mathscr{A} \underset{\mathfrak{A}}{=} \mathsf{T} \text{ and } \mathscr{B} \underset{\mathfrak{A}}{=} \mathsf{T}\right)$$

$$\Rightarrow \left\{\alpha \mid \mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \in D \text{ and } \left\{\alpha \mid \mathscr{B} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \in D,$$

whence

$$\left\{\alpha \mid \mathscr{A} \,\&\, \mathscr{B} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} = \left\{\alpha \mid \mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \cap \left\{\alpha \mid \mathscr{B} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \in D,$$

and therefore $\mathscr{A} \,\&\, \mathscr{B}$ is a filtered formula. In a similar way, one proves that the formula $(\exists x_1)\mathscr{A}$ is filtered.

Lemma 8. *If the formula $\mathscr{A}(x_1, \ldots, x_n)$ defines a predicate which is filtered in an ultrafilter D, then the formula $\neg\, \mathscr{A}$ also defines a filtered predicate in D.*

Since for any subset S of I,

$$S \in D \Leftrightarrow S' \notin D \qquad (S' = I - S),$$

it follows that

$$\left\{\alpha \mid \neg\, \mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \in D \Leftrightarrow \left\{\alpha \mid \mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \notin D \Leftrightarrow \neg\, \mathscr{A} \underset{\mathfrak{A}}{=} \mathsf{T}.$$

Lemmas 7 and 8 immediately imply the following fundamental result.

Theorem 9. *Let $\mathscr{A}(R_1, \ldots, R_s;\ x_1, \ldots, x_n)$ be an arbitrary first-order formula, where $R_1, \ldots, R_s$ are predicates and $x_1, \ldots, x_n$ are the object symbols occurring in this formula. If $R_1, \ldots, R_s$ are filtered predicates in an ultrafilter D, then the predicate defined by $\mathscr{A}$ is also filtered in D.*

Indeed, $\mathscr{A}$ is equivalent to a formula obtainable from the atomic formulas $R_i(x_{i_1}, \ldots, x_{i_p})$ with the use of &, $\exists$ and $\neg$. By hypothesis, the atomic formulas define filtered predicates in D. By Lemmas 7 and 8, the operations &, $\exists$ and $\neg$, when applied to predicates filtered in the ultrafilter D, result in filtered predicates in D. Therefore, formula $\mathscr{A}$ defines a filtered predicate in D.

It was noted above that in a filtered product of algebraic systems of given signature Ω, all the predicates and operations of the signature are filtered. Therefore, Theorem 7 immediately yields the following.

Corollary 10. *Suppose $\mathscr{A}(x_1, \ldots, x_n)$ is an arbitrary first-order formula of given signature Ω with free variables $x_1, \ldots, x_n$. Let $\langle I, D\rangle$ be an arbitrary ultrafilter and let $\alpha \to \mathfrak{A}_\alpha$ be a mapping of I into the class of algebraic systems of signature Ω. Then for any $a_1, \ldots, a_n$ in $\prod \mathfrak{A}_\alpha$ we have*

$$\mathscr{A}(a_1 D, \ldots, a_n D) = \mathsf{T} \Leftrightarrow \{\alpha \mid \mathscr{A}(a_1^\alpha, \ldots, a_n^\alpha) = \mathsf{T}\} \in D.$$

In particular, a closed formula $\mathscr{A}$ of signature Ω is true in a filtered product of algebraic systems $\mathfrak{A}_\alpha$ ($\alpha \in I$) modulo an ultrafilter $\langle I, D\rangle$ if and only if the set of indices of the factors in which $\mathscr{A}$ is true belongs to the ultrafilter D, i.e.

$$\mathscr{A} \underset{\mathfrak{A}}{=} \mathsf{T} \Leftrightarrow \left\{\alpha \mid \mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\right\} \in D \qquad (\mathfrak{A} = \prod \mathfrak{A}_\alpha / D). \tag{10}$$

A filtered product of systems modulo an ultrafilter is called an *ultraproduct*. For a given filter D over some set I, all the sets belonging to the filter are often said to be big (relative to D) or to contain almost all the elements of I. In this terminology, (10) may be expressed in the form: *a closed first-order formula is true in an ultraproduct if and only if it is true in almost all its factors.*

In particular, if a closed first-order formula is true in each algebraic system $\mathfrak{A}_\alpha$ ($\alpha \in I$) ,then it is true in any ultraproduct of these systems.

For example, by Section 3.3, a group is an algebraic system with signature $\{\cdot, ^{-1}\}$ in which the formula

$$(\forall xyz)\left(x(yz) = (xy)z \,\&\, x(yy^{-1}) = x \,\&\, (y^{-1}y)x = x\right)$$

is true. Using Corollary 10, we conclude that every ultraproduct of groups is a group.

The group property of being abelian is also expressible by a first-order formula: $(\forall xy)\,(xy = yx)$. Therefore, an ultraproduct of abelian groups is an abelian group.

A group is called *divisible* if the formulas

$$(\forall x)\,(\exists y)\,(y^n = x) \qquad (n = 2, 3, \ldots). \tag{11}$$

hold in it. Consequently, an ultraproduct of divisible groups is a divisible group.

Denote by Z_{p_i} the cyclic group of prime order p_i ($i = 0, 1, 2, \ldots$; $p_0 = 2, p_1 = 3, \ldots$), and let D be a nonprincipal ultrafilter over the set N of all natural numbers. Let us try to determine the structure of the group

$$\mathfrak{G} = \prod Z_{p_i} / D.$$

First of all, the Z_{p_i} are abelian groups of order tending to infinity. By Corollary 10 and Theorem 3 of Section 8.5, $\mathfrak{G}$ is an infinite abelian group having cardinality of the continuum. The formula

$$(\forall x)\,(x^{n+1} = x \to x = xx^{-1}) \qquad (n \geqq 1)$$

is true in each factor starting with Z_{p_n}. Therefore it is true in $\mathfrak{G}$, i.e. $\mathfrak{G}$ contains no elements of finite order. Analogously, formula (11) for each $n \geqq 1$ is true in all factors starting with Z_{p_n}. Therefore it is also true in $\mathfrak{G}$; that is, $\mathfrak{G}$ is divisible.

If $\prod \mathfrak{G}_\alpha$ is the direct product of groups $\mathfrak{G}_\alpha$, then the subgroup consisting of elements g which have only a finite number of projections $g\pi_\alpha$ distinct from the identity is called the *weak direct product* of the given groups. According to the well-known theorem from the theory of abelian groups, every divisible abelian group without elements of finite order is isomorphic to a weak direct product of a suitable number of groups isomorphic to the additive group of rational numbers. Thus, the group of interest to us is isomorphic to the weak direct product of a continuum of copies of the additive group of rational numbers.

A class $\mathfrak{K}$ of algebraic systems of signature Ω is called *axiomatizable* (*finitely axiomatizable*) or *elementary* if there exists a collection $\mathfrak{S}$ (finite collection $\mathfrak{S}$) of closed APC-formulas of signature Ω such that $\mathfrak{K}$ consists of exactly the algebraic systems of signature Ω in which all the formulas of $\mathfrak{S}$ are true (cf. Section 6.4).

We say that a class $\mathfrak{K}$ of algebraic systems of signature Ω is *closed with respect to ultraproducts* (briefly, *ultraclosed*) if for each set I, each ultrafilter D over I and all algebraic systems $\mathfrak{A}_i$ $(i \in I)$ chosen from $\mathfrak{K}$, the ultraproduct $\prod \mathfrak{A}_i/D$ $(i \in I)$ belongs to $\mathfrak{K}$.

A class $\mathfrak{K}$ of systems of signature Ω is called *abstract* if whenever $\mathfrak{K}$ contains a system $\mathfrak{A}$, it also contains all systems of signature Ω isomorphic to it.

From Theorem 1 of Section 6.3, it follows that *every axiomatizable class of systems is abstract.* From Corollary 10 we obtain: *every axiomatizable class is also ultraclosed.*

8.3 Applications of Ultraproducts. As a first application of the above theory of ultraproducts we shall prove the following important theorem.

Theorem 1. (Compactness Theorem or Local Theorem for First-Order Languages). *Let $\mathfrak{S}$ be an infinite collection of closed first-order formulas of signature Ω. If every finite subset of $\mathfrak{S}$ is satisfiable, then the entire collection $\mathfrak{S}$ is satisfiable.*

A collection of closed formulas of given signature Ω is called *satisfiable* or *consistent* if there exists an algebraic system of signature Ω

in which each formula of the collection is true. Denote by I the set of all finite nonempty subsets of $\mathfrak{S}$. Let $\Phi_\alpha (\alpha \in I)$ denote the conjunction of all formulas belonging to α. By hypothesis to each $\alpha \in I$, there exists an algebraic system $\mathfrak{M}_\alpha$ of signature Ω in which the formula Φ_α is true. Introduce the subsets $I_\xi = \{\alpha \in I \mid \Phi_\xi \underset{\mathfrak{M}_\alpha}{=} \mathsf{T}\}$.

Since the intersection $I_\xi \cap \ldots . I_\zeta$ is equal to

$$\{\alpha \mid \Phi_\xi \,\&\, \ldots \,\&\, \Phi_\zeta \underset{\mathfrak{M}_\alpha}{=} \mathsf{T}\} = I_{\xi \cup \ldots \cup \zeta} \neq \varnothing,$$

then by Theorem 3 of Section 8.1, the collection of subsets of the form I_ξ can be extended to an ultrafilter D over I. Consider the ultraproduct

$$\mathfrak{M} = \prod \mathfrak{M}_\alpha / D \qquad (\alpha \in I).$$

By Corollary 10 of the previous subsection

$$\Phi_\alpha \underset{\mathfrak{M}}{=} \mathsf{T} \Leftrightarrow \{i \mid \Phi_\alpha \underset{\mathfrak{M}_i}{=} \mathsf{T}\} \in D \Leftrightarrow I_\alpha \in D.$$

Thus, all the formulas Φ_α, and hence all the formulas of $\mathfrak{S}$, are true in $\mathfrak{M}$. Consequently, the collection $\mathfrak{S}$ is satisfiable.

For a finite or countable signature, Theorem 1 follows from the Gödel Completeness Theorem. An independent proof for an arbitrary signature was indicated by the author in the paper [38]. It was also pointed out how to use the general compactness theorem to obtain specific algebraic results. These ideas were developed by the author in [42]. There the compactness theorem led to the solution of certain problems in general group theory of interest at that time to the specialists in that field. Other important applications were found by A. Robinson (see [62]) and a number of other authors.

In Section 7.2, we considered various criteria for classes of algebraic systems to be universally axiomatizable. The theory of ultraproducts allows us to give other criteria. They rest on the following result.

Theorem 2. *If every finite reduct $\langle A_\alpha, \Omega_\beta \rangle$ of every finite submodel $\langle A_\alpha, \Omega \rangle$ of an algebraic system $\mathfrak{A} = \langle A, \Omega \rangle$ is embeddable in some system $\mathfrak{M}_{\alpha\beta} = \langle M_{\alpha\beta}, \Omega \rangle$, then $\mathfrak{A}$ is embeddable in some ultraproduct of the systems $M_{\alpha\beta}$.*

Denote by $f_{\alpha\beta}$ the isomorphism of the model $\langle A_\alpha, \Omega_\beta \rangle$ into the system $\mathfrak{M}_{\alpha\beta}$, which exists by hypothesis. The conclusion of Theorem 2 is nontrivial only if A or Ω is infinite, which we shall assume. Denote by I the collection of all submodels $\mathfrak{A}_{\alpha\beta} = \langle A_\alpha, \Omega_\beta \rangle$ and introduce the subcollections

$$I_{\xi\eta} = \{\mathfrak{A}_{\alpha\beta} \mid A_\xi \subseteq A_\alpha \,\&\, \Omega_\eta \subseteq \Omega_\beta\}.$$

Clearly, the intersection of all subcollections $I_{\xi\eta}$ is empty, while each finite intersection $I_{\xi_1\eta_1} \cap \ldots \cap I_{\xi_s\eta_s}$ is nonempty since it coincides with $I_{\xi\eta}$, where

$$A_\xi = \bigcup A_{\xi_i}, \quad \Omega_\eta = \bigcup \Omega_{\eta_i}.$$

Now extend the system of subcollections $I_{\xi\eta}$ to some ultrafilter D over I and consider the ultraproduct

$$\mathfrak{M} = \prod \mathfrak{M}_{\alpha\beta}/D.$$

We select an arbitrary sequence $\{d_i \mid i \in I\}$ in $\prod \mathfrak{M}_{\alpha\beta}$ and for any $a \in \mathfrak{A}$ we denote by a^*D the sequence $\{a^i \mid i \in I\}$, where

$$a^i = \begin{cases} f_{\alpha\beta}(a) & (a \in A_\alpha), \\ d_i & (a \notin A_\alpha). \end{cases}$$

We shall show that the mapping $a \to a^*$ is an embedding of $\mathfrak{A}$ into $\mathfrak{M}$. We may assume that the equality sign is included in Ω. Let P be an n-ary predicate symbol in Ω and $a_1, \ldots, a_n$ arbitrary elements of $\mathfrak{A}$. Set $A_\xi = \{a_1, \ldots, a_n\}$ and $\Omega_\eta = \{P\}$. Then if $P(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$,

$$\{i \in I \mid P(a_1^i, \ldots, a_n^i) = \mathsf{T}\} \supseteq I_{\xi\eta},$$

and hence $P(a_1^*D, \ldots, a_n^*D) = \mathsf{T}$ in $\mathfrak{M}$. If $P(a_1, \ldots, a_n)$ is false in $\mathfrak{A}$, then

$$\{i \in I \mid P(a_1^i, \ldots, a_n^i) = \mathsf{F}\} \supseteq I_{\xi\eta}$$

and hence $P(a_1^*D, \ldots, a_n^*D) = \mathsf{F}$. Consequently, the mapping $a \to a^*$ is an isomorphism of $\mathfrak{A}$ into $\mathfrak{M}$ and Theorem 2 is proved.

According to Section 7.2, an algebraic system $\mathfrak{A}$ is *locally embeddable* in a class of systems $\mathfrak{K}$ if every finite reduct of every finite submodel of $\mathfrak{A}$ is embeddable in some system of $\mathfrak{K}$.

Since any ultraproduct of systems from an axiomatizable class belongs to that class, Theorem 2 immediately implies the following statement.

Corollary 3. *If an algebraic system $\mathfrak{A}$ is locally embeddable in an axiomatizable class $\mathfrak{K}$, then $\mathfrak{A}$ is isomorphically embeddable in some system of $\mathfrak{K}$.*

This corollary can be viewed as one of the necessary properties for a class of systems to be axiomatizable.

It is also a direct trivial consequence of Theorem 1. As a necessary property for axiomatizability it was explicitly formulated by Henkin in [19] and is now sometimes called the Henkin property.

It will be recalled that a subclass $\mathfrak{L}$ of a class $\mathfrak{K}$ of systems is *universally axiomatizable* within $\mathfrak{K}$ (see Section 7.2) if $\mathfrak{L}$ consists of all

$\mathfrak{K}$-systems that satisfy a fixed collection of closed $\forall$-formulas. By the Tarski-Łos theorem, a sublass $\mathfrak{L}$ is universally axiomatizable within $\mathfrak{K}$ if and only if any $\mathfrak{K}$-system $\mathfrak{A}$ being locally embeddable in $\mathfrak{L}$ implies that $\mathfrak{A} \in \mathfrak{L}$. By means of Theorem 2, the Tarski-Łos conditions can be made to take on an especially simple form if one supposes that $\mathfrak{L}$ is an ultraclosed class (i.e., closed with respect to ultraproducts) or is, in particular, axiomatizable.

A subclass $\mathfrak{L}$ of a class $\mathfrak{K}$ is *hereditary in* $\mathfrak{K}$ (Section 7.2) if every $\mathfrak{K}$-subsystem of each $\mathfrak{L}$-system is an $\mathfrak{L}$-system. The Tarski-Łos conditions imply that the condition of $\mathfrak{L}$ being hereditary in $\mathfrak{K}$ is necessary for $\mathfrak{L}$ to be universally axiomatizable within $\mathfrak{K}$.

Corollary 4. *An ultraclosed abstract subclass $\mathfrak{L}$ is universally axiomatizable within an ultraclosed class $\mathfrak{K}$ of algebraic systems if and only if $\mathfrak{L}$ is hereditary in $\mathfrak{K}$.*

In particular, an (absolutely) axiomatizable subclass $\mathfrak{L}$ is universally axiomatizable within an axiomatizable class $\mathfrak{K}$ if and only if $\mathfrak{L}$ is hereditary in $\mathfrak{K}$.

We have already remarked that necessity is obvious. We shall prove sufficiency. Suppose an abstract subclass $\mathfrak{L}$ is ultraclosed and hereditary in an ultraclosed class $\mathfrak{K}$. Suppose some $\mathfrak{K}$-system $\mathfrak{A}$ is locally embeddable in $\mathfrak{L}$. Then by Theorem 2, $\mathfrak{A}$ is embeddable in an ultraproduct $\mathfrak{M}$ of $\mathfrak{L}$-systems. Since $\mathfrak{L}$ is ultraclosed, it follows that $\mathfrak{M} \in \mathfrak{L}$; since the subclass $\mathfrak{L}$ is hereditary, $\mathfrak{A} \in \mathfrak{L}$. The Tarski-Łos conditions are satisfied and therefore $\mathfrak{L}$ is universally axiomatizable in $\mathfrak{K}$.

Corollary 5. *A class $\mathfrak{L}$ of algebraic systems is universally axiomatizable if and only if*

i) *$\mathfrak{L}$ is ultraclosed;*

ii) *$\mathfrak{L}$ is (absolutely) hereditary;*

iii) *$\mathfrak{L}$ is abstract.*

This is a special case of Corollary 4 obtainable by assuming that $\mathfrak{K}$ is the class of all algebraic systems of given signature. Corollary 5 shows, for instance, that a class of groups is universally axiomatizable if it is ultraclosed, contains the subgroups of all its groups, and is abstract.

These results are applicable to the embedding problem, often encountered in algebra in various forms.

Theorem 6. *Let $\mathfrak{K}$ and $\mathfrak{K}_1$ be ultraclosed abstract classes of algebraic systems of signature Ω and Ω_1, respectively, with $\Omega \subseteq \Omega_1$. Then the class $\mathfrak{L}$ of $\mathfrak{K}$-systems which are embeddable in some $\mathfrak{K}_1$-system is universally axiomatizable within $\mathfrak{K}$ (or is empty).*

Indeed, if $\mathfrak{A}$ is a $\mathfrak{K}$-subsystem of an $\mathfrak{L}$-system $\mathfrak{B}$, then $\mathfrak{B}$ is embeddable in some $\mathfrak{K}_1$-system $\mathfrak{M}$. But then $\mathfrak{A}$ is also embeddable in $\mathfrak{M}$, i.e. $\mathfrak{A} \in \mathfrak{L}$ and the class $\mathfrak{L}$ is hereditary in $\mathfrak{K}$. Clearly, since the classes $\mathfrak{K}$ and $\mathfrak{K}_1$ are ultraclosed, $\mathfrak{L}$ is ultraclosed. Thus the conditions of Corollary 5 are satisfied and hence the sublcass $\mathfrak{L}$ is universally axiomatizable within $\mathfrak{K}$.

Let $\mathfrak{K}$ be an ultraclosed abstract class of algebras of signature Ω and $\mathfrak{K}_1$ an ultraclosed abstract class of algebraic systems of signature $\Omega_1 \supseteq \Omega$. In particular, the classes $\mathfrak{K}$ and $\mathfrak{K}_1$ may be axiomatizable. The problem of embedding $\mathfrak{K}$-algebras in $\mathfrak{K}_1$-systems consists in finding conditions which must be satisfied by those $\mathfrak{K}$-algebras which are embeddable in a $\mathfrak{K}_1$-system. Theorem 6 asserts that such conditions can be put in the form of $\forall$-formulas. But every $\forall$-formula of signature Ω is a conjunction of formulas of the form

$$\text{a) } (\forall x_1 \dots x_n)\,(F_1 \neq G_1 \vee \dots \vee F_m \neq G_m),$$

$$\text{b) } (\forall x_1 \dots x_n)\,(F_1 = G_1),$$

$$\text{c) } (\forall x_1 \dots x_n)\,(F_1 = G_1 \,\&\, \dots \,\&\, F_m = G_m \to F_{m+1} = G_{m+1}),$$

$$\text{d) } (\forall x_1 \dots x_n)\,(F_1 = G_1 \,\&\, \dots \,\&\, F_m = G_m \to F_{m+1} = G_{m+1} \vee \dots \vee F_s = G_s),$$

where $F_1, G_1, \dots, F_s, G_s$ are terms of signature Ω in the variables $x_1, \dots, x_n$. In principle then, the answer can always be represented by formulas of types a)—d) whose satisfiability in the $\mathfrak{K}$-algebra $\mathfrak{A}$ is a necessary and sufficient condition for the embeddability of $\mathfrak{A}$ in some $\mathfrak{K}_1$-system. Of course, while knowing the form of the answer, we must in each specific case find the desired formulas and investigate their mutual dependence.

As an example, consider the well-known problem of embedding associative rings into associative division rings, i.e. associative rings satisfying the axiom *)

$$(\exists x)\,(x \neq 0) \,\&\, (\forall a)\,\big(a \neq 0 \to (\forall b)\,(\exists xy)\,(ax = ya = b)\big). \qquad (1)$$

Rings embeddable in division rings obviously have no zero divisors (Section 4.1), and therefore we may seek conditions for embedding rings without zero divisors (the class $\mathfrak{K}$) into associative division rings (the class $\mathfrak{K}_1$). Both classes are axiomatizable, and therefore, the class $\mathfrak{L}$ of embeddable rings can be characterized within $\mathfrak{K}$ by axioms of the form a)—d). A one-element ring belongs to the class $\mathfrak{L}$. But in a one-element ring, formulas of type a) are false. Thus, the desired conditions must be of the form b)—d).

*) See Problems and Complements to Section 4.

Since rings in the class $\mathfrak{L}$ have no zero divisors,

$$F_{m+1} = G_{m+1} \vee \ldots \vee F_s = G_s$$

is equivalent in $\mathfrak{L}$ to the formula

$$(F_{m+1} - G_{m+1}) \ldots (F_s - G_s) = 0.$$

In other words, formulas of type d) can be reduced to the form c). Formulas of type b) may be replaced by equivalent formulas of the form

$$x_1 = x_1 \to F_1 = G_1,$$

which are also of type c). Thus, we have shown that the conditions for embedding rings without zero divisors into division rings can be represented by a collection of conditions of the form

$$\Phi_1 = 0 \,\&\, \ldots \,\&\, \Phi_s = 0 \to \Phi = 0, \tag{2}$$

where $\Phi_1, \ldots, \Phi_s$ and Φ are polynominals in the (noncommutative) variables $x_1, \ldots, x_n$.

According to a classical theorem in ring theory, every commutative, associative ring without zero divisors is embeddable in a field. In 1937, it was also shown (Mal'cev [39]) that there exist associative, noncommutative rings without zero divisors which are not embeddable in any associative division ring. At the present time, various sets of necessary conditions for embeddability of the form (2) are known. Some sufficient conditions are also known. However, explicit necessary and sufficient conditions have not yet been found. In particular, we do not know whether such conditions are expressible as a finite family of formulas of the form (2).

8.4 Conditionally Filtered Formulas. According to Section 7.5, a closed first-order formula $\mathscr{A}$ of signature Ω is *multiplicatively stable* if the satisfiability of $\mathscr{A}$ in the algebraic systems $\mathfrak{A}_\alpha$ ($\alpha \in I$) of signature Ω implies that $\mathscr{A}$ is satisfiable in the direct product $\prod \mathfrak{A}_\alpha$. Comparing the notions of multiplicative stability and conditional filtration introduced in Section 8.2, we immediately see that a conditionally filtered closed formula is multiplicatively stable.

In Section 7.5, we noted that identities and conditional identities are multiplicatively stable. Apparently, the question of finding the most general form of multiplicatively stable formulas has not been completely settled. However, in 1951, Horn [22] found a simple class of multiplicatively stable formulas which attracted the attention of many

researchers. They are now known as Horn formulas. These formulas were studied in Section 7.5. We state their definition in the following form.

A Horn formula $\mathscr{A}$ with free object variables $x_1, \ldots, x_m$ is a formula of the form

$$(Q_1 x_{m+1}) \ldots (Q_{n-m} x_n)\ (\mathscr{A}_1 \,\&\, \ldots \,\&\, \mathscr{A}_s) \qquad (Q_i = \forall, \exists). \tag{1}$$

Here each $\mathscr{A}_i$ is either a disjunction

$$\neg\, \mathscr{P}_{i1} \vee \neg\, \mathscr{P}_{i2} \vee \ldots \vee \neg\, \mathscr{P}_{ik},$$

or an implication

$$\mathscr{P}_{i1} \,\&\, \mathscr{P}_{i2} \,\&\, \ldots \,\&\, \mathscr{P}_{ik} \to \mathscr{P}_{i0},$$

constructed from formulas $\mathscr{P}_{ij}$ of the form $f = g$ or of the form $R(f_1, \ldots, f_t)$ ($R \in \Omega$; $f, g, f_1, \ldots, f_t$ are terms in the variables $x_1, \ldots, x_n$).

The identities and conditional identities mentioned in Section 7.5 are Horn formulas. It is also clear that a conjunction of Horn formulas is equivalent to a Horn formula. In particular, the condition that a predicate $R(x_1, \ldots, x_n, x)$ is a function is describable by the Horn formula

$$(\forall x_1 \ldots x_n y z)\ (\exists x)\ (R(x_1, \ldots, x_n, x) \,\&\, \big(R(x_1, \ldots, x_n, y) \,\&\, R(x_1, \ldots, x_n, z) \to y = z\big)).$$

We also note that every Horn formula containing terms is easily converted into a Horn formula (1) in which all the subformulas $\mathscr{P}_{ij}$ are atomic formulas of the form $R(x_{i_1}, \ldots, x_{i_p})$ ($i_1, \ldots, i_p = 1, \ldots n$), where R belongs to $\{\Omega_p, =\}$ or corresponds to an operator symbol in Ω. To do this, we merely have to replace each term by its expression given in Section 7.1.

Horn [22] showed that all closed Horn formulas are multiplicatively stable (see also Section 7.5). We shall now prove the following stronger assertion due to Chang and Morel [10].

Theorem 1. *All Horn formulas are conditionally filtered in any filter. In particular, if a closed Horn formula $\mathscr{A}$ is true in each algebraic system $\mathfrak{A}_\alpha$ ($\alpha \in I$) of fixed signature, then $\mathscr{A}$ is also true in the filtered product $\prod \mathfrak{A}_\alpha/D$ of these systems modulo any filter D over I.*

We first prove the following lemma.

Lemma 2. *If $\mathscr{P}_1, \ldots, \mathscr{P}_k$ are formulas of signature Ω with free object variables $x_1, \ldots, x_n$ which are filtered in the filtered product $\mathfrak{A} = \prod \mathfrak{A}_\alpha/D$ of algebraic systems $\mathfrak{A}_\alpha$ of signature Ω and the formula $\mathscr{P}_0(x_1, \ldots, x_n)$ is*

conditionally filtered in this product, then the formulas

$$\neg \mathscr{P}_1 \vee \ldots \vee \neg \mathscr{P}_k, \quad \mathscr{P}_1 \& \ldots \& \mathscr{P}_k \to \mathscr{P}_0$$

are conditionally filtered in it.

By Lemma 5 of Section 8.2, the formula $\mathscr{P} = \mathscr{P}_1 \& \ldots \& \mathscr{P}_k$ is filtered in the given filtered product, that is

$$\mathscr{P} \underset{\mathfrak{A}}{=} \mathsf{T} \Leftrightarrow \{\alpha \mid \mathscr{P} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D,$$

and consequently,

$$\{\alpha \mid \neg \mathscr{P} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D \Leftrightarrow \{\alpha \mid \mathscr{P} \underset{\mathfrak{A}_\alpha}{=} \mathsf{F}\} \in D \Rightarrow \mathscr{P} \underset{\mathfrak{A}}{=} \mathsf{F}.$$

Therefore, the formula $\neg \mathscr{P} = \neg \mathscr{P}_1 \vee \ldots \vee \neg \mathscr{P}_k$ defines a conditionally filtered predicate.

Let us show that the formula $\mathscr{P} \to \mathscr{P}_0$ is also conditionally filtered. Suppose that $\{\alpha \mid \mathscr{P} \to \mathscr{P}_0 \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D$. We must check that $\mathscr{P} \underset{\mathfrak{A}}{=} \mathsf{T}$ implies $\mathscr{P}_0 \underset{\mathfrak{A}}{=} \mathsf{T}$. Since the predicate $\mathscr{P}$ is filtered, $\mathscr{P} \underset{\mathfrak{A}}{=} \mathsf{T}$ implies $\{\alpha \mid \mathscr{P} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D$. But

$$\{\alpha \mid \mathscr{P}_0 \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \supseteq \{\alpha \mid \mathscr{P} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \cap \{\alpha \mid \mathscr{P} \to \mathscr{P}_0 \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\} \in D,$$

and hence $\mathscr{P}_0 \underset{\mathfrak{A}}{=} \mathsf{T}$.

We now turn to the proof of Theorem 1. We may assume that the formula $\mathscr{A}(x_1, \ldots, x_m)$ has the form (1) and that the $\mathscr{P}_{ij}$ are atomic. Atomic formulas are filtered in any filter. By Lemma 2, we conclude that the subformulas $\mathscr{A}_1, \ldots, \mathscr{A}_s$ occurring in $\mathscr{A}$ are conditionally filtered. Now applying Lemma 6 of Section 8.2, we see that $\mathscr{A}_1 \& \ldots \& \mathscr{A}_s$ and hence $\mathscr{A}$ is conditionally filtered.

From the definition of conditional filtration, it is immediately seen that if a formula $\mathscr{A}$ is conditionally filtered in a product, then every formula equivalent to $\mathscr{A}$ is also filtered in this product. Therefore, Theorem 1 implies that every formula equivalent to a Horn formula is conditionally filtered in any filtered product.

Assuming the generalized continuum hypothesis,*) Keisler [26] proved that the converse is also true: if a formula is conditionally filtered in any filtered product, then it is equivalent to a Horn formula.

The question arises whether the conditional filtration of formulas in filters of some special type is sufficient for the formula to be equivalent to a Horn formula. In particular, there was a time when it was conjectured in the literature that conditional filtration of formulas in

*) The assumption that for every infinite cardinal $\mathfrak{c}$ there is no cardinal between $\mathfrak{c}$ and $2^{\mathfrak{c}}$ is called the generalized continuum hypothesis [*Ed.*].

direct products was sufficient for reducibility of the formula to a Horn formula. However, Chang and Morel [10] found several counterexamples. One such counterexample is the formula

$$(\exists a b)\,(\forall x)\,(a \neq b \,\&\, ax = a \,\&\, bx = a \vee bx = b). \tag{2}$$

This formula is conditionally filtered in all direct products. Indeed, suppose formula (2) is true in each algebra $\mathfrak{A}_\alpha = \langle A_\alpha, \cdot\rangle$ $(\alpha \in I)$. Let $\mathfrak{A}$ denote the direct product of these algebras. By hypothesis, each $\mathfrak{A}_\alpha$ contains elements $a_\alpha \neq b_\alpha$ such that for each $x_\alpha \in A_\alpha$

$$a_\alpha x_\alpha = a_\alpha \,\&\, b_\alpha x_\alpha = a_\alpha \vee b_\alpha x_\alpha = b_\alpha. \tag{3}$$

Let a and b be elements of the algebra $\mathfrak{A}$ whose projections a^α and b^α are given by

$$a^\alpha = a_\alpha \qquad (\alpha \in I), \tag{4}$$

$$b^\alpha = \begin{cases} a_\alpha & (\alpha \neq \lambda), \\ b_\alpha & (\alpha = \lambda), \end{cases} \tag{5}$$

where λ is some fixed index in I. Since $a_\lambda \neq b_\lambda$, $a \neq b$. From (4) it follows that for each $x \in \mathfrak{A}$ and each $\alpha \in I$, $a^\alpha x^\alpha = a^\alpha$, that is, $ax = a$. From (5) and (3) it follows that

$$b^\alpha x^\alpha = b^\alpha = a^\alpha \qquad (\alpha \neq \lambda),$$

$$b^\lambda x^\lambda = a^\lambda \vee b^\lambda x^\lambda = b^\lambda.$$

If $b^\lambda x^\lambda = a^\lambda$, then $bx = a$; if $b^\lambda x^\lambda = b^\lambda$, then $bx = b$, and hence formula (2) holds in $\mathfrak{A}$.

It remains to show that formula (2) is not equivalent to any Horn formula. If it were equivalent to a Horn formula, then by Theorem 1 it would be conditionally filtered in any filtered product. We shall show that this is not the case. Denote by D a Fréchet filter over the set N $(N = \{0, 1, 2, \ldots\})$; that is, D is a collection of sets of natural numbers with finite complements. With each $n \in N$, we associate the same algebra $\mathfrak{A}_n = \langle\{0, 1\}, \cdot\rangle$, where $\cdot$ denotes the usual multiplication of natural numbers. Clearly, formula (2) holds in $\mathfrak{A}_n$. We shall show that not formula (2) is true in the filtered product $\prod \mathfrak{A}_n/D$ but rather its negation. We write the negation in the form

$$(\forall a b)\,\big(a \neq b \,\&\, (\forall x)\,(ax = a) \rightarrow (\exists y)\,(by \neq a \,\&\, by \neq b)\big). \tag{6}$$

Formula (6) is true in the filtered product $\prod \mathfrak{A}_n/D$ if and only if the formula

$$(\forall a b)\,\big(a \neq_D b \,\&\, (\forall x)\,(ax \equiv_D a) \rightarrow (\exists y)\,(by \neq_D a \,\&\, by \neq_D b)\big) \tag{7}$$

is true in the usual product $\prod \mathfrak{A}_n$. Elements a and b of the direct product can be written in the form of sequences

$$a = \langle a_0, a_1, \ldots\rangle; \quad b = \langle b_0, b_1, \ldots\rangle \quad (a_i, b_i = 0, 1).$$

The condition $a \equiv_D b$ means that $a_i \neq b_i$ for $i \in I_1$, where I_1 is an infinite set of natural numbers. Since $(\forall x)\,(ax \equiv_D a)$, we must have $a \equiv_D ax$ for $x = \langle 0, 0, \ldots\rangle$, i.e., $a_i = 0$ for all sufficiently large i. Therefore $b_i = 1$ for sufficiently large $i \in I_1$. Split set I_1 into two infinite sets I_2 and I_3 such that $y_i = 1$ for $i \in I_3$ and $y_i = 0$ for $i \notin I_3$. Then $b_i y_i = y_i$ for sufficiently large i and hence $by \equiv_D y$. On the other hand, $y \not\equiv_D a$ and $y \not\equiv_D b$ since $y_i = 0$ and $b_i = 1$ for sufficiently large $i \in I_2$. Thus, formula (7) is true in $\prod \mathfrak{A}_n$.

8.5 Cardinality of Ultraproducts. Since ultrafilters are contructed with the help of the axiom of choice, it is generally not an easy task to determine the exact cardinality of an ultraproduct. The results below are concerned with bounds for the cardinalities of various ultraproducts.

Theorem 1. *If the cardinality of each factor in an ultraproduct does not exceed a fixed natural number n, then the cardinality of the ultraproduct also does not exceed n.*

By hypothesis, the formula

$$(\forall x_1 \ldots x_{n+1})\,(x_1 = x_2 \vee x_1 = x_3 \vee \ldots \vee x_n = x_{n+1})$$

is true in each factor of the ultraproduct. By the fundamental theorem on ultraproducts, this formula is also true in the ultraproduct.

Theorem 2. *If in the filtered product $\prod \mathfrak{A}_\alpha/D$ $(\alpha \in I)$, D is not a principal filter and if for each natural number n the number of factors of cardinality n is finite, then the filtered product has infinite cardinality. Any filtered product of infinite systems is infinite.*

Suppose that the filter D in the filtered product $\prod \mathfrak{A}_\alpha/D$ is not principal. This implies that all the sets in D are infinite. Denote by D_1 the collection of sets obtained from those in D by deleting a finite number of elements. Clearly, D_1 is again a nonprincipal filter containing all cofinite sets. $D \subseteq D_1$ implies that the filtered product $\prod \mathfrak{A}_\alpha/D_1$ is the homomorphic image of the product $\prod \mathfrak{A}_\alpha/D$. Therefore, Theorem 2 will have been established if the former product is infinite. Consider the formula

$$(\exists x_1 \ldots x_n)\,(x_1 \neq x_2 \,\&\, x_1 \neq x_3 \,\&\, \ldots \,\&\, x_{n-1} \neq x_n), \qquad (1)$$

which asserts that the cardinality of the system is not less than n. This is a Horn formula and therefore is conditionally filtered in any filtered product. By hypothesis, formula (1) can be false in only a finite number of the factors $\mathfrak{A}_\alpha$. Every cofinite set belongs to D_1. Hence, formula (1) is true in $\prod \mathfrak{A}_\alpha/D_1$ for any n.

Before formulating more delicate theorems, we shall make several obvious remarks. According to the definition, the elements of the filtered product $\prod \langle A_\alpha, \Omega\rangle/D$ are the equivalence classes of functions in the cartesian product $\prod A_\alpha$ modulo the filter D. In the definition of the equivalence modulo the filter, no symbols of the signature are involved. Consequently, the cardinality of the filtered product $\prod \mathfrak{A}_\alpha/D$ is completely determined by the cardinalities of the factors $\mathfrak{A}_\alpha$ and the nature of the filter D. Hence, we can denote the cardinality of the product $\prod \mathfrak{A}_\alpha/D$ by $\prod |\mathfrak{A}_\alpha|/D$. Moreover, if $A_\alpha \subseteq B_\alpha$ $(\alpha \in I)$, then $\prod A_\alpha/D \subseteq \prod B_\alpha/D$ and consequently, the cardinality of the filtered product depends monotonically on the cardinality of each factor.

If $D \subseteq D_1$, then $\prod \mathfrak{A}_\alpha/D_1$ is a homomorphic image of the product $\prod \mathfrak{A}_\alpha/D$. Therefore,

$$|\prod \mathfrak{A}_\alpha/D_1| \leqq |\prod \mathfrak{A}_\alpha/D|. \tag{2}$$

In particular, for any filter D

$$|\prod \mathfrak{A}_\alpha/D| \leqq \prod |\mathfrak{A}_\alpha|. \tag{3}$$

Inequality (3) furnishes an upper bound for the cardinality of a filtered product which is sometimes also the lower bound.

By Theorem 3 of Section 8.2, if the intersection M of all sets of the filter D is nonempty and does not belong to D, then

$$\prod \mathfrak{A}_\alpha/D \cong \prod_{\mu \in M} \mathfrak{A}_\mu \cdot \prod_{\gamma \in M'} \mathfrak{A}_\gamma/D_0,$$

where $M' = I - M$, and the filter D_0 consists of the intersections of M' with the sets of the filter D. The intersection of all the sets of D_0 is empty and therefore D_0 contains all the sets with finite complement in M'. Thus, in studying the cardinalities of filtered products we may restrict our attention to those filters consisting of sets all with finite complements.

A filter $\langle I, D\rangle$ is called *homogeneous* if the cardinality of each set in D is equal to the cardinality of I.

Suppose that the filter $\langle I, D\rangle$ is not homogeneous. Then D contains some set M whose cardinality is less than $|I|$.

By formula (5) of Section 8.2,

$$\prod \mathfrak{A}_\alpha/D \cong \prod \mathfrak{A}_\mu/D_1 \qquad (\mu \in M),$$

where D_1 is the filter formed by the intersections of M with the sets of the filter D. Thus, the study of the cardinalities of arbitrary filtered products is reduced to the study of the cardinalities of filtered products modulo homogeneous filters containing sets all with finite complements.

Let α be a cardinal number. A filter D is called *α-complete* if the intersection of any α sets in D belongs to D. In particular, a *countably complete* filter D is one in which the intersection of a countable number of arbitrary sets in D again belongs to D.

Suppose we are given a filter D over a countable set I and that every set with finite complement belongs to D. Then D cannot be countably complete. In fact, successively deleting one element at a time from I, we obtain a countable number of sets belonging to the filter whose intersection is empty. If the filter D were countably complete, then it would contain the empty set which is impossible.

Every nonprincipal ultrafilter contains all sets with finite complement. Thus, in particular, there does not exist a countably complete nonprincipal ultrafilter over a countable set.

Theorem 3. *Let $\prod \mathfrak{A}_\alpha/D$ $(\alpha \in I)$ be an ultraproduct with a nonprincipal ultrafilter D which is not countably complete. Suppose that for each natural number n*

$$\{\alpha \mid |\mathfrak{A}_\alpha| = n\} \notin D. \tag{4}$$

Then the cardinality of the ultraproduct is at least $2^{\aleph_0}$, i.e., at least the cardinality of the continuum.

In particular, *if the factors $\mathfrak{A}_\alpha$ are finite or countable, if I is a collection of natural numbers, if D is a nonprincipal ultrafilter over I and condition (4) is fulfilled, then the cardinality of the ultraproduct $\prod \mathfrak{A}_\alpha/D$ is equal to $2^{\aleph_0}$.*

The proof uses the following lemma which is of interest in itself. Other proofs are given in the paper by Frayne, Morel and Scott [15] and in Cohn's book [3].

Lemma 4. *Consider a set of unary predicate symbols P_σ $(\sigma \in \Omega)$, indexed by all possible sequences $\sigma = \langle \sigma_1, \sigma_2, \ldots, \sigma_m, \ldots \rangle$, $\sigma_i \in \{0,1\}$. Consider the collection $\mathfrak{S}$ of all formulas of the form*

$$(\exists x) P_\sigma(x), \tag{5}$$

$$(\exists x_1 \ldots x_n)\,(x_1 \neq x_2 \,\&\, x_1 \neq x_3 \,\&\, \ldots \,\&\, x_{n-1} \neq x_n) \rightarrow (\forall x)\left(\neg P_\lambda(x) \vee \neg P_\mu(x)\right) \tag{6}$$

$$(n = 2^m;\ \langle \lambda_1, \ldots, \lambda_m \rangle \neq \langle \mu_1, \ldots, \mu_m \rangle;\ m = 1, 2, \ldots).$$

Every infinite model $\mathfrak{M}$ for $\mathfrak{S}$ contains a submodel of cardinality $2^{\aleph_0}$. To each natural number $s \geqq 1$, there exists a model $\mathfrak{M}_s$ for $\mathfrak{S}$ containing 2^s elements.

The first assertion is obvious since by axioms (5), to each $\sigma \in \Omega$ there exists $a_\sigma \in \mathfrak{M}$ for which $P_\sigma(a_\sigma) = \mathsf{T}$. Since $\mathfrak{M}$ has an infinite number of elements, we have $a_\lambda \neq a_\mu$ for $\lambda \neq \mu$ and hence $\{a_\sigma \mid \sigma \in \Omega\}$ is a submodel of cardinality $2^{\aleph_0}$.

Suppose s is a positive integer. Denote by M_s the set of all sequences

$$\beta = \langle \beta_1, \beta_2, \ldots, \beta_s \rangle \qquad (\beta_i \in \{0, 1\}).$$

For $\beta \in M_s$, $\sigma \in \Omega$, set $P_\sigma(\beta) = \mathsf{T}$ if β is an initial segment of the sequence σ and set $P_\sigma(\beta) = \mathsf{F}$ otherwise. The resulting model $\langle M_s, \Omega \rangle$ will be denoted by $\mathfrak{M}_s$. It has 2^s elements. The axioms (5) are true in $\mathfrak{M}_s$: for x, take $[\sigma]_s$ the initial segment of length s of sequence σ. The axioms of the form (6) for which $m > s$ are true in $\mathfrak{M}_s$ since the antecedent of the implication (6) is false in $\mathfrak{M}_s$. The axioms of the form (6) for which $m \leqq s$ are true in $\mathfrak{M}_s$ since if $P_\lambda(x)$ and $P_\mu(x)$ are true for some $x \in M_s$, then $[\lambda]_s = [\mu]_s = x$ which contradicts the assumption $[\lambda]_m \neq [\mu]_m$.

Thus, the collection $\mathfrak{S}$ has models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ of unbounded finite order, models of the cardinality of the continuum and larger, and no models of countable cardinality (or other cardinalities less than $2^{\aleph_0}$, should such cardinals exist).

Now let us return to the proof of Theorem 3. Suppose

$$\{\alpha \mid |\mathfrak{A}_\alpha| \geqq \aleph_0\} = J.$$

Since D is an ultrafilter, either $J' \in D$ or $J \in D$. By Theorem 2 of Section 8.2, we may restrict our attention to the cases

i) $J' = I$, ii) $J = I$.

Case i). All the factors $\mathfrak{A}_\alpha$ are finite and their cardinalities are unbounded above. From each factor $\mathfrak{A}_\alpha$ whose number of elements $|\mathfrak{A}_\alpha|$ satisfies $2^s \leqq |\mathfrak{A}_\alpha| < 2^{s+1}$, we select a subset $\mathfrak{B}_\alpha$ containing exactly 2^s elements. We define in $\mathfrak{B}_\alpha$ the predicates P_σ of Lemma 4 so that $\mathfrak{B}_\alpha$ becomes a model isomorphic to the model $\mathfrak{M}_s$ mentioned in Lemma 4. Since in each factor $\mathfrak{B}_\alpha$ of the ultraproduct

$$\mathfrak{B} = \prod \mathfrak{B}_\alpha / D \qquad (\alpha \in I)$$

all the axioms of the collection $\mathfrak{S}$ are true, $\mathfrak{S}$ is true also in $\mathfrak{B}$. Choose a formula $\mathscr{A}_s$ of the form (1) for which $n = 2^s$. Since

$$\left\{\alpha \mid \mathscr{A}_s \underset{\mathfrak{B}_\alpha}{=} \mathsf{T}\right\} = \{\alpha \mid |\mathfrak{A}_\alpha| \geqq 2^s\} \in D,$$

$\mathscr{A}_s$ is true in $\mathfrak{B}$; that is, $\mathfrak{B}$ is an infinite model for $\mathfrak{S}$. By Lemma 4, this implies that $|\mathfrak{B}| \geqq 2^{\aleph_0}$ and therefore, $|\prod \mathfrak{A}_\alpha / D| \geqq 2^{\aleph_0}$.

Case ii). All the factors $\mathfrak{A}_\alpha$ are infinite. By hypothesis, the filter D is not countably complete. This means that there are sets U_i $(i = 1, 2, \ldots)$ in D such that $U = \bigcap U_i$ does not belong to D. Instead of all sets U_i consider now the intersections $U_1 \cap U_2 \cap \ldots \cap U_i$ keeping only those which are pairwise distinct. This yields a decreasing sequence

$$V_1 \supset V_2 \supset \ldots \supset V \qquad (V_i \in D,\ V = \bigcap V_i \notin D).$$

Let D_0 denote the ultrafilter formed by the intersections of the set V' of D with the sets in D. Putting $V_i - V = W_i$, we have

$$\prod_{\alpha \in I} \mathfrak{A}_\alpha/D \cong \prod \mathfrak{A}_\lambda/D_0 \qquad (\lambda \in V'),$$

$$W_0 = V' \supset W_1 \supset W_2 \supset \ldots \qquad (W_i \in D_0,\ \bigcap W_i = \varnothing).$$

In each system $\mathfrak{A}_\lambda$ whose index λ belongs to the difference $W_s - W_{s+1}$ $(s = 0, 1, \ldots)$, we choose a subset $\mathfrak{B}_\lambda$ of 2^s elements and we define $\mathfrak{B}_\lambda$ the predicate P_σ of Lemma 4 so that the collection $\mathfrak{S}$ is satisfiable in $\mathfrak{B}_\lambda$. Since in each model $\mathfrak{B}_\lambda$ all the formulas of $\mathfrak{S}$ are true, $\mathfrak{S}$ is also satisfiable in the ultraproduct

$$\mathfrak{B} = \prod \mathfrak{B}_\lambda/D_0.$$

It remains to show that the model $\mathfrak{B}$ is infinite. Consider again the axiom $\mathscr{A}_s$ of the form (1), where $n = 2^s$. Since

$$\{\lambda \mid \mathscr{A}_s \underset{\mathfrak{B}_\lambda}{=} \mathsf{T}\} = W_s \in D_0,$$

$\mathscr{A}_s$ is true in $\mathfrak{B}$ for any s, i.e. the model $\mathfrak{B}$ is infinite. By Lemma 4, $|\mathfrak{B}| \geqq 2^{\aleph_0}$.

Theorem 3 furnishes precise information about the cardinality of a filtered product for filters over a countable set I. But it does not reveal how the cardinality of ultraproducts grows with the growth of $|I|$. This gap is partially filled by the next result.

Theorem 5 (Frayne, Morel, and Scott [15]). *To every infinite I, there exists a filter D such that for every filter $D_1 \supseteq D$ and every infinite system $\mathfrak{A}$*

$$|\mathfrak{A}^I/D_1| \geqq 2^{|I|}.$$

The proof of this theorem is based on the following lemma of Chang and Keisler on the embeddability of direct products.

Lemma 6. *Let $\langle J_k \mid k \in K\rangle$ be a sequence of subsets of I containing all nonempty finite subsets of I and let $\langle \mathfrak{A}_\alpha \mid \alpha \in I\rangle$ be a sequence of algebraic*

systems $\mathfrak{A}_\alpha$ of given signature Ω. Then there is a filter $\langle K, D\rangle$ such that to any filter $D_1 \supseteq D$ over K, there exists an isomorphic embedding

$$\prod_{\alpha\in I} \mathfrak{A}_\alpha \to \prod_{k\in K} \left(\prod_{\alpha\in J_k} \mathfrak{A}_\alpha\right)/D_1. \tag{7}$$

For each $\alpha \in I$, set

$$L_\alpha = \{k \mid \alpha \in J_k\}.$$

For arbitrary $\alpha, \ldots, \gamma \in I$

$$L_\alpha \cap \ldots \cap L_\gamma = \{k \mid \alpha, \ldots, \gamma \in J_k\} \neq \varnothing,$$

and therefore, the collection of all supersets of the intersections $L_\alpha \cap \ldots \cap L_\gamma$ in K forms a filter D over K. This filter satisfies the requirements of Lemma 6. Indeed, with each function $f \in \prod \mathfrak{A}_\alpha$ we associate the function $f_p \in \prod_{k\in K} \prod_{\alpha\in J_k} \mathfrak{A}_\alpha$ defined by

$$f_p(k)\,(\alpha) = f(\alpha) \qquad (\alpha \in J_k).$$

It remains to be shown only that $g \neq f$ $(g, f \in \prod \mathfrak{A}_\alpha)$ implies $g_p \not\equiv_{D1} f_p$. By hypothesis $g(\alpha) \neq f(\alpha)$ for some $\alpha \in I$ and hence $\alpha \in J_k \Rightarrow g_p(k) \neq f_p(k)$; that is, $\{k \mid g_p(k) \neq f_p(k)\} \supseteq L_\alpha$. Hence,

$$\{k \mid g_p(k) \neq f_p(k)\} \in D_1$$

and consequently, $g_p \equiv_{D1} f_p$.

Theorem 5 is an immediate consequence of Lemma 6. Indeed, since I is infinite, there is a one-to-one mapping $\alpha \to J_\alpha$ of I onto the collection of all its nonempty finite subsets J_α. Hence, applying Lemma 6 to the filtered power $\mathfrak{A}^I/D_1$, we may set $K = I$. From (7) we obtain

$$|\mathfrak{A}^I| \leqq \left|\prod_{\alpha\in I} (\mathfrak{A}^{J_\alpha})/D_1\right|$$

or

$$2^{|I|} \leqq |\mathfrak{A}^I| \leqq \prod_{\alpha\in I} |\mathfrak{A}^{J_\alpha}|/D_1. \tag{8}$$

By hypothesis, the system $\mathfrak{A}$ is infinite and hence $|\mathfrak{A}^{J_\alpha}| = |\mathfrak{A}|$ (Section 2.6). The replacement of $|\mathfrak{A}^{J_\alpha}|$ by $|\mathfrak{A}|$ in (8) completes the proof of Theorem 5.

As already mentioned, Theorems 1—5 furnish only general bounds for the cardinalities of filtered products. However they are sufficient for the main applications. More precise bounds are given in the complements to this section.

8.6 Regular Products. Regular products were discovered almost at the same times as filtered products and they have also come to play a significant role in the study of problems couched in first-order language.

Consider two signatures Ω_0 and Ω_1. Suppose that Ω_1 contains the set-theoretic operations $\cup$, $\cap$, $^-$, the predicate symbol $\subseteq$ for set-theoretic inclusion, the symbol $\varnothing$ for the empty set and possibly other predicate and operator symbols.

Let $\mathscr{P}(I)$ be the set of all subsets of I. An algebraic system $\mathfrak{S}$ of signature Ω_1 will be called an *algebra of subsets* if its carrier is $\mathscr{P}(I)$ for some I and the signature symbols $\cup$, $\cap$, $^-$ are made to correspond in $\mathfrak{S}$ to set-theoretic union, intersection and complementation of subsets of I, the symbol $\subseteq$ to set-theoretic inclusion in $\mathscr{P}(I)$ and $\varnothing$ to the empty subset.

Suppose that to each element α of a nonempty set I, there corresponds an algebraic system $\mathfrak{A}_\alpha = \langle A_\alpha, \Omega_0 \rangle$ of signature Ω_0. Consider the cartesian product $M = \prod A_\alpha$ $(\alpha \in I)$. Suppose that $\mathscr{A}(x_1, \ldots, x_n)$ is an APC-formula of signature Ω_0. For $f_1, \ldots, f_n \in M$, let $K_{\mathscr{A}}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n)$ denote the set

$$\{\alpha \in I \mid \mathscr{A}(f_1^\alpha, \ldots, f_n^\alpha) \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\}.$$

In particular, if $\mathscr{A}$ is a closed formula of signature Ω_0, then

$$K_{\mathscr{A}}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}} = \{\alpha \in I \mid \mathscr{A} \underset{\mathfrak{A}_\alpha}{=} \mathsf{T}\}.$$

The following exemplify properties immediately implied by this definition:

$$K_{\neg \mathscr{A}}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n) = I - K_{\mathscr{A}}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n);$$

$$K_{\mathscr{A}_1 \& \mathscr{A}_2}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n) = K_{\mathscr{A}_1}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n) \cup K_{\mathscr{A}_2}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n);$$

$$K_{\mathscr{A}_1 \vee \mathscr{A}_2}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n) = K_{\mathscr{A}_1}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n) \cup K_{\mathscr{A}_2}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_1, \ldots, f_n);$$

$$K_{(\exists x_1)\mathscr{A}}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_2, \ldots, f_n) = \bigcup_{g \in M} K_{\mathscr{A}}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(g, f_2, \ldots, f_n).$$

Consider all possible sequences $\xi = \langle \mathscr{B}; \mathscr{A}_0, \ldots, \mathscr{A}_m \rangle$ where $\mathscr{B}$ is an APC-formula of signature Ω_1 and $\mathscr{A}_0, \ldots, \mathscr{A}_m$ are APC-formulas of signature Ω_0. A sequence ξ will be called *admissible* if all the free object variables in $\mathscr{B}$ are from the set $\{x_0, \ldots, x_m\}$. The only free variables of ξ will be those object variables which have a free occurence in at least one of the formulas $\mathscr{A}_0, \ldots, \mathscr{A}_m$.

A *standard admissible* sequence ξ will be one in which the free variables of ξ are precisely $x_0, \ldots, x_k$ for some natural number k. The sequence ξ is called *splitting* if the formulas

$$\mathscr{A}_0 \vee \ldots \vee \mathscr{A}_m, \quad \neg (\mathscr{A}_i \& \mathscr{A}_j) \; (i < j \leq m)$$

are identically true.

Let $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1 \rangle$ be an algebra of subsets. With each standard admissible sequence

$$\xi = \langle \mathscr{B}(x_0, \ldots, x_m); \mathscr{A}_0(x_0, \ldots, x_k), \ldots, \mathscr{A}_m(x_0, \ldots, x_k) \rangle,$$

we associate a $(k+1)$-ary predicate $Q_\xi^{\mathfrak{S}}$ in M by setting for $f_0, \ldots, f_k \in M$

$$Q_\xi^{\mathfrak{S}}(f_0, \ldots, f_k) = \mathsf{T} \Leftrightarrow \mathscr{B}\left(K_{\mathscr{A}_0}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_0, \ldots, f_k), \ldots, K_{\mathscr{A}_m}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_0, \ldots, f_k)\right) = \mathsf{T}.$$

We use the transfinite numbers less than β to index all the standard admissible sequences. Let ξ_α be the standard admissible sequence indexed by the number α.

A *complete regular product* $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i\,(i \in I)$ of algebraic systems $\mathfrak{A}_i\,(i \in I)$ over the algebra of subsets $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1 \rangle$ is the model $\langle M, \{Q_{\xi_\alpha}^{\mathfrak{S}}\} \rangle$ $(\alpha < \beta)$.

A *regular product* of algebraic systems $\mathfrak{A}_i\,(i \in I)$ over the algebra of subsets $\mathfrak{S}$ is any reduct of the model $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i(i \in I)$.

If all the models $\mathfrak{A}_i$ are the same model $\mathfrak{A}$, then we write $\mathfrak{A}^{\mathfrak{S}}$ in place of $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i(i \in I)$. The model $\mathfrak{A}^{\mathfrak{S}}$ is called the complete regular power of the algebraic system $\mathfrak{A}$ over the algebra of subsets $\mathfrak{S}$. The reducts of $\mathfrak{A}^{\mathfrak{S}}$ are called regular powers of the model $\mathfrak{A}$ over the algebra of subsets $\mathfrak{S}$.

Let $\Omega = \Omega(\Omega_0, \Omega_1) = \{Q_{\xi_\alpha}\}\,(\alpha < \beta)$. With each Q_{ξ_α}, we associate a predicate $Q_{\xi_\alpha}^{\mathfrak{S}}$. $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i(i \in I)$ can then be viewed as a model of signature Ω.

The following are some examples of regular products.

Example 1. Let $\Omega_1 = \langle \cup, \cap, {}^-, \subseteq, \varnothing \rangle$. With each n-ary predicate symbol P in Ω_0, we associate the standard admissible sequence

$$\xi^{(P)} = \langle \bar{x}_0 = \varnothing\,;\, P(x_0, \ldots, x_{n-1}) \rangle.$$

With each n-ary operator symbol F from Ω_0, we associate the standard admissible sequence

$$\xi^{(F)} = \langle \bar{x}_0 = \varnothing\,;\, F(x_0, \ldots, x_{n-1}) = x_n \rangle.$$

Now for $f_0, \ldots, f_{n-1} \in M$,

$$Q_{\xi^{(P)}}(f_0, \ldots, f_{n-1}) = \mathsf{T} \Leftrightarrow I - K_{P(x_0, \ldots, x_{n-1})}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_0, \ldots, f_{n-1}) = \varnothing$$

$$\Leftrightarrow K_{P(x_0, \ldots, x_{n-1})}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_0, \ldots, f_{n-1}) = I.$$

Analogously,

$$Q_{\xi^{(F)}}(f_0, \ldots, f_{n-1}, f_n) = \mathsf{T} \Leftrightarrow K_{F(x_0, \ldots, x_{n-1}) = x_n}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}}(f_0, \ldots, f_{n-1}, f_n) = I.$$

Therefore, if $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1 \rangle$ is an algebra of subsets and $\Omega' = \langle Q_{\xi^{(F)}}, Q_{\xi^{(P)}} \mid F, P \in \Omega_0 \rangle$, then the Ω'-reduct of $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i(i \in I)$ is a model representing the direct product of the algebraic systems $\mathfrak{A}_i(i \in I)$ (of course, up to renaming the basic operations and predicates).

Example 2. Let $\Omega_1 = \langle \cup, \cap, ^-, \subseteq, \varnothing, \mathrm{Fin}\rangle$ where Fin is the symbol for a unary predicate and let $\Omega_0 = \langle \cdot, ^{-1}, e\rangle$, where $\cdot$ is a binary operation symbol, $^{-1}$ is a unary operation, and e is the symbol for a distinguished element. Let $\mathfrak{A}_i (i \in I)$ be a group (with $\cdot$ its multiplication, $^{-1}$ its inverse operation, and e its identity), and let $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1\rangle$ be an algebra of subsets such that $\mathrm{Fin}(x) = \mathsf{T}$ if and only if x is a finite subset of I.

If

$$\xi_0 = \langle \mathrm{Fin}\,(x_0); x_0 \neq e\rangle; \xi_1 = \langle \bar{x}_0 = \varnothing; x_0 \cdot x_1 = x_2\rangle;$$

$$\xi_2 = \langle \bar{x}_0 = \varnothing; x_0^{-1} = x_1\rangle; \quad \xi_3 = \langle \bar{x}_0 = \varnothing; x_0 = e\rangle,$$

then, as already remarked, the $\langle Q_{\xi_1}, Q_{\xi_2}, Q_{\xi_3}\rangle$-reduct of $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_i (i \in I)$ is a model representing the direct product of groups $\mathfrak{A}_i (i \in I)$. Also, it is easy to see that $Q_{\xi 0}(f) = \mathsf{T}$ if and only if $f^\alpha \neq e$ only for finitely many α in I.

Example 3. Let $\Omega_1 = \langle \cup, \cap, -, \subseteq, \varnothing, D\rangle$, where D is a unary predicate symbol. For simplicity we shall assume that Ω_0 contains only predicate symbols. Let $\mathfrak{A}_\alpha (\alpha \in I)$ be models of signature Ω_0.

Consider the algebra of subsets $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1\rangle$, in which the set of all $x \in \mathscr{P}(I)$ for which $D(x) = \mathsf{T}$ forms a filter over I.

For each n-ary predicate $P \in \Omega_0$, consider the standard admissible sequence $\xi^{(P)} = \langle D(x_0); P(x_0, \ldots, x_{n-1})\rangle$. Then for $f_0, \ldots, f_{n-1} \in M$

$$Q^{\mathfrak{S}}_{\xi^{(P)}}(f_0, \ldots, f_{n-1}) = \mathsf{T} \Leftrightarrow D\big(K^{\{\mathfrak{A}_\alpha | \alpha \in I\}}_{P(x_0, \ldots, x_{n-1})}(f_0, \ldots, f_{n-1})\big) = \mathsf{T}.$$

In other words

$$Q^{\mathfrak{S}}_{\xi^{(P)}}(f_0, \ldots, f_{n-1}) = \mathsf{T} \Leftrightarrow P(f_0 D, \ldots, f_{n-1} D) \underset{\Pi \mathfrak{A}_\alpha / D}{=} \mathsf{T}.$$

Now consider the sequence $\xi = \langle D(x_0); x_0 = x_1\rangle$ and let f_0 and $f_1 \in M$. Clearly, $Q_\xi{}^{\mathfrak{S}}(f_0, f_1)$ is true if and only if $f_0 D = f_1 D$.

It is easy to verify that $Q_\xi{}^{\mathfrak{S}}$ is a congruence relation on the model $\langle M, \{Q^{\mathfrak{S}}_{\xi^{(P)}}\}\rangle$ $(P \in \Omega_0)$ and that the quotient model of $\langle M, \{Q^{\mathfrak{S}}_{\xi^{(P)}}\}\rangle$ $(P \in \Omega_0)$ modulo $Q_\xi{}^{\mathfrak{S}}$ is $\prod \mathfrak{A}_\alpha / D (\alpha \in I)$. This was actually done in Section 8.2.

The examples given show that the notion of regular product encompasses many algebraic constructions. Therefore, important applications have been found for the following proposition.

Theorem 1. *There is an effective procedure for associating with each APC-formula $\Gamma(x_{i_0}, \ldots, x_{i_k})$ of signature $\Omega = \Omega(\Omega_0, \Omega_1)$, where $x_{i_0}, \ldots, x_{i_k}$ are pairwise distinct, a standard admissible splitting sequence*

$$\xi = \langle \mathscr{B}, \mathscr{A}_0(x_0, \ldots, x_k), \ldots, \mathscr{A}_m(x_0, \ldots, x_k)\rangle$$

such that for any nonempty family $\mathfrak{A}_\alpha = \langle A_\alpha, \Omega_0\rangle$ $(\alpha \in I)$ *of algebraic systems of signature* Ω_0 *and for any algebra of subsets* $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1\rangle$*)

$$(\forall x_{i_0} \dots x_{i_k})\left(\Gamma(x_{i_0}, \dots, x_{i_k}) \leftrightarrow Q_\xi^{\mathfrak{S}}(x_{i_0}, \dots, x_{i_k})\right) \underset{\mathscr{P}^{\mathfrak{S}}\mathfrak{U}_\alpha}{=} \mathsf{T}. \tag{1}$$

In particular if Γ *is a closed formula of signature* Ω*, then* $\mathscr{A}_0, \dots, \mathscr{A}_m$ *are closed formulas of signature* Ω_0 *and*

$$\Gamma \underset{\mathscr{P}^{\mathfrak{S}}\mathfrak{U}_\alpha}{=} \mathsf{T} \Leftrightarrow Q_\xi^{\mathfrak{S}} \underset{\mathscr{P}^{\mathfrak{S}}\mathfrak{U}_\alpha}{=} \mathsf{T}. \tag{2}$$

The proof will be by induction on the number of logical symbols $\vee$, $\neg$ and $\exists$ occurring in the formula Γ, it being assumed that there are no occurrences of the symbols $\rightarrow$, $\forall$ and $\&$ in Γ.

We first prove that with each standard admissible sequence $\xi' = \langle \mathscr{B}'; \mathscr{A}_0', \dots, \mathscr{A}'_m\rangle$, we can effectively associate a standard admissible splitting sequence ξ with the same free variables such that for any algebra of subsets $\mathfrak{S} = \langle \mathscr{P}(I), \Omega\rangle$ and for any family $\mathfrak{A}_\alpha = \langle A_\alpha, \Omega_0\rangle$ $(\alpha \in I)$ of algebraic systems of signature Ω_0, the predicates $Q_\xi^{\mathfrak{S}}$ and $Q_{\xi'}^{\mathfrak{S}}$, coincide on $M = \prod A_\alpha$ $(\alpha \in I)$.

Let $m = 2^{m'+1}$, and let $r_0, \dots, r_m$ be the sequence of all subsets of $\{0, 1, \dots, m'\}$. For each natural number $k \leqq m$ let

$$\mathscr{A}_k = (\bigwedge_{j \in r_k} \mathscr{A}_j') \,\&\, (\bigwedge_{j \in \{0,1,\dots,m'\} - r_k} \neg \mathscr{A}_j')$$

(here, the conjunction of an empty set of formulas is taken to be identically true). For each natural number $l \leqq m'$, let

$$s_l = \{k \mid k \leqq m \,\&\, l \in r_k\},$$

and let

$$\mathscr{B} = \mathscr{B}'\left(\bigcup_{k \in s_0} x_k, \dots, \bigcup_{k \in s_{m'}} x_k\right),$$

where for a finite set s of natural numbers, $\bigcup_{i \in s} x_i$ is $\varnothing$ if s is empty or is $(\bigcup_{i \in s'} x_i) \cup x_k$ if k is the largest number in s and $s' = s - \{k\}$. It can be directly verified that $\xi = \langle \mathscr{B}; \mathscr{A}_0, \dots, \mathscr{A}_m\rangle$ is a standard admissible splitting sequence and that the predicates $Q_\xi^{\mathfrak{S}}$ and $Q_{\xi'}^{\mathfrak{S}}$ coincide in M.

We return to the proof of the theorem. First we consider the case when Γ is an atomic formula.

If Γ is $Q_\xi(x_{i_0}, \dots, x_{i_k})$, it is sufficient to refer to the remark with which we began the proof the theorem.

---

*) $\mathscr{A} \leftrightarrow \mathscr{B}$ is being used as an abbreviation for the formula $(\mathscr{A} \rightarrow \mathscr{B})$ $\&$ $(\mathscr{B} \rightarrow \mathscr{A})$

If Γ is $Q_{\xi'}(x_{j_0}, \ldots, x_{j_s})$, where $j_0, \ldots, j_s$ are certain numbers among $i_0, \ldots, i_k$, then we set $\xi' = \langle \mathscr{B}; \mathscr{A}_0', \ldots, \mathscr{A}_m' \rangle$. Suppose that $\mathscr{A}_i$ is obtained from $\mathscr{A}_i'$ by replacing $x_{m'}$ with x_m for each pair (m, m') such that x_{j_m} is $x_{i_{m'}}$ $(m = 1, \ldots, s;\ m' = 1, \ldots, k)$. It is directly verified that the sequence $\xi = \langle \mathscr{B}; \mathscr{A}_0, \ldots, \mathscr{A}_m \rangle$ satisfies condition (1); this sequence ξ can be transformed into a splitting sequence.

If Γ is $x_{i_0} = x_{i_1}$, take ξ to be the sequence $\langle \bar{x}_0 = \varnothing;\ x_0 = x_1,\ x_0 \neq x_1 \rangle$ and if Γ is $x_{i_0} = x_{i_0}$, take ξ to be the sequence $\langle \bar{x}_0 = \varnothing;\ x_0 = x_0,\ x_0 \neq x_0 \rangle$.

If Γ_1 is $Q_{\xi_1}(x_{j_0}^1, \ldots, x_{j_{s_1}}^1)$, where $\xi_1 = \langle \mathscr{B}_1; \mathscr{A}_0^1, \ldots, \mathscr{A}_{m_1}^1 \rangle$, and Γ_2 is $Q_{\xi_2}(x j_0^2, \ldots, x_{j_{s_2}}^2)$, where $\xi_2 = \langle \mathscr{B}_2; \mathscr{A}_0^2, \ldots, \mathscr{A}_{m_2}^2 \rangle$, we first note that $Q_{\xi_1}(x_0, \ldots, x_{s_1}) \vee Q_{\xi_2}(x_{s_1+1}, \ldots, x_{s_1+s_2+1})$ is $Q_\xi(x_0, \ldots, x_{s_1+s_2+1})$ for a suitable sequence ξ. Such a ξ may be taken to be

$$\langle \mathscr{B}_1(x_0, \ldots, x_{m_1}) \vee \mathscr{B}_2(x_{m_1+1}, \ldots, x_{m_1+m_2+1});\ \mathscr{A}_0^1,$$
$$\ldots, \mathscr{A}_{m_1}^1, \mathscr{A}_0^2(x_{s_1+1}, \ldots, x_{s_1+s_2+1}),$$
$$\ldots, \mathscr{A}_{m_2}^2(x_{s_1+1}, \ldots, x_{s_1+s_2+1}) \rangle.$$

Now $\Gamma_1 \vee \Gamma_2$ is $Q_\xi(x j_0^1, \ldots, x_{j_{s_1}}^1, x_{j_0}^2, \ldots, x_{j_{s_2}}^2)$ and the matter is reduced to a previous case.

If the formula Γ is $Q_{\xi'}(x_{j_0}, \ldots, x_{j_s})$, then $\neg\, \Gamma$ is $Q_\xi(x_{j_0}, \ldots, x_{j_s})$, where $\xi = \langle \neg\, \mathscr{B}; \mathscr{A}_0, \ldots, \mathscr{A}_m \rangle$ if $\xi' = \langle \mathscr{B}; \mathscr{A}_0, \ldots, \mathscr{A}_m \rangle$.

There remains the case where Γ is the formula $(\exists x_{i_{k+1}}) Q_{\xi'}(x_{i_0}, x_{i_1}, \ldots, x_{i_k}, x_{i_{k+1}})$. Suppose ξ' is a splitting sequence and $\xi' = \langle \mathscr{B}'; \mathscr{A}_0', \ldots, \mathscr{A}_m' \rangle$. Suppose $\mathscr{A}_i$ is $(\exists x_{i_{k+1}})\, \mathscr{A}_i'$, and $\mathscr{B}$ is

$$(\exists y_0) \ldots (\exists y_m) \Big(\bigcup_{j \leqq m} y_j = \overline{\varnothing}\ \&\ \bigwedge_{i<j \leqq m} (y_i \cap y_j = \varnothing)$$
$$\&\ \bigwedge_{j \leqq m} (y_j \subseteq x_j)\ \&\ \mathscr{B}'(y_0, \ldots, y_m) \Big).$$

Suppose that $\xi = \langle \mathscr{B}; \mathscr{A}_0, \ldots, \mathscr{A}_m \rangle$. Let us show that ξ satisfies condition (1).

Suppose $f_0, \ldots, f_k \in M$ and $Q_\xi(f_0, \ldots, f_k) = \mathsf{T}$. Then there are pairwise disjoint subsets $y_0, \ldots, y_m$ in I such that

$$\mathscr{B}'(y_0, \ldots, y_m) \underset{\mathfrak{S}}{=} \mathsf{T},$$
$$y_j \subseteq K^{\{\mathfrak{A}_\alpha | \alpha \in I\}}_{(\exists x_{i_{k+1}}) \mathscr{A}_j'}(f_0, \ldots, f_k), \qquad y_0 \cup \ldots \cup y_m = I.$$

If $\alpha \in I$, then there is one and only one natural number $j \leqq m$ such that $\alpha \in y_j$. Hence, there is an $f_{k+1}^\alpha \in A_\alpha$ such that $\mathscr{A}_j'(f_0, \ldots, f_k, f_{k+1}) = \mathsf{T}$.

Let us show that

$$Q_{\xi'}(f_0, \ldots, f_k, f_{k+1}) \underset{\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_\alpha}{=} \mathsf{T}.$$

We must show that

$$\mathscr{B}'\left(K^{\{\mathfrak{A}_\alpha|\alpha\in I\}}_{\mathscr{A}_0'}(f_0, \ldots, f_k, f_{k+1}), \ldots, K^{\{\mathfrak{A}_\alpha|\alpha\in I\}}_{\mathscr{A}_m'}(f_0, \ldots, f_k, f_{k+1})\right) \underset{\mathfrak{S}}{=} \mathsf{T}.$$

But $K_j = K^{\{\mathfrak{A}_\alpha|\alpha\in I\}}_{\mathscr{A}_j'}(f_0, \ldots, f_k, f_{k+1}) \supseteq y_j$, the sets $K_0, \ldots, K_m$ are pairwise disjoint and their union is I. Hence, $K_j = y_j$. Now $\mathscr{B}'(K_0, \ldots, K_m) \underset{\mathfrak{S}}{=} \mathsf{T}$.

Conversely, suppose $f_0, \ldots, f_k \in M$ and $\Gamma(f_0, \ldots, f_k) = \mathsf{T}$. Then there is an $f_{k+1} \in M$ such that $Q_{\xi'}(f_0, \ldots, f_k, f_{k+1}) = \mathsf{T}$. Suppose $y_j = K^{\{\mathfrak{A}_\alpha|\alpha\in I\}}_{\mathscr{A}_j'}(f_0, \ldots, f_{k+1})$. Then by definition of $Q_{\xi'}$, $y_0, \ldots, y_m$ are pairwise disjoint, their union is I and $\mathscr{B}'(y_0, \ldots, y_m) = \mathsf{T}$. Moreover, $y_i \subseteq K^{\{\mathfrak{A}_\alpha|\alpha\in I\}}_{(\exists x_{k+1})\mathscr{A}_i'}(f_0, \ldots, f_k)$. This shows that $Q_\xi(f_0, \ldots, f_k) = \mathsf{T}$.

The sequence ξ constructed need not be splitting but it can be converted into a splitting sequence. Theorem 1 is proved.

Recall that the elementary theory Th ($\mathfrak{K}$) for a class $\mathfrak{K}$ of algebraic systems of signature σ is the collection of all closed APC-formulas of signature σ which are true in every system in $\mathfrak{K}$. The elementary theory for $\mathfrak{K}$ is called *decidable* if there is a decision procedure which determines whether or not an arbitrary closed APC-formula of signature σ is an element of Th ($\mathfrak{K}$).

A simple consequence of Theorem 1 is the next proposition.

Theorem 2. *Let $\mathfrak{K}$ be a class of algebraic systems of signature Ω_0, $\mathfrak{L}$ a class of algebras of subsets of signature Ω_1, and $\mathscr{P}_{\mathfrak{K},\mathfrak{L}}$ the class of all models $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_\alpha (\alpha \in I)$ such that $\mathfrak{A}_\alpha \in \mathfrak{K}$ for all $\alpha \in I$, $\mathfrak{S} \in \mathfrak{L}$, and let the carrier of the algebra of subsets $\mathfrak{S}$ be $\mathscr{P}(I)$. If* Th ($\mathfrak{K}$) *and* Th ($\mathfrak{L}$) *are decidable theories, then* Th ($\mathscr{P}_{\mathfrak{K},\mathfrak{L}}$) *is also decidable.*

Let Γ be a closed formula of signature $\Omega = \Omega(\Omega_0, \Omega_1)$. We shall find a standard admissible splitting sequence $\xi = \langle \mathscr{B}; \mathscr{A}_0, \ldots, \mathscr{A}_m \rangle$ which satisfies condition (2).

For each $\mathscr{A}_i$, we determine whether or not $\neg \mathscr{A}_i \in$ Th ($\mathfrak{K}$). Let t be the set of all natural numbers $j \leqq m$ for which $\neg \mathscr{A}_j \in$ Th ($\mathfrak{K}$). Let

$$\Psi = (\forall x_0 \ldots x_m)\Big(\big(\bigcup_{j\leqq m} x_j' = \overline{\varnothing}$$

$$\& \bigwedge_{i<j\leqq m}(x_j \cap x_i = \varnothing) \& \bigwedge_{j\in t} x_j = \varnothing) \to \mathscr{B}(x_0, \ldots, x_m)\Big).$$

We can determine whether or not $\Psi \in$ Th ($\mathfrak{L}$). We wish to prove that

$$\Gamma \in \text{Th}\,(\mathscr{P}_{\mathfrak{K},\mathfrak{L}}) \Leftrightarrow \Psi \in \text{Th}\,(\mathfrak{L}).$$

If $\Psi \in \mathrm{Th}(\mathfrak{L})$, then of course, $\Gamma \in \mathrm{Th}(\mathscr{P}_{\mathfrak{L},\mathfrak{L}})$. Suppose that $\Gamma \in \mathrm{Th}(\mathscr{P}_{\mathfrak{L},\mathfrak{L}})$ and $\Psi \notin \mathrm{Th}(\mathfrak{L})$. Then there are $\mathfrak{S} = \langle \mathscr{P}(I), \Omega_1 \rangle \in \mathfrak{L}$ and pairwise disjoint $x_0, \ldots, x_m$ whose union is I such that $x_j = \varnothing$ for $j \in t$ and $\mathscr{B}(x_0, \ldots, x_m)$ is false in $\mathfrak{S}$. For $\alpha \in I$, choose x_j such that $\alpha \in x_j$ and choose $\mathfrak{A}_\alpha$ to be a system from $\mathfrak{K}$ in which $\mathscr{A}_i$ is true. Since $j \notin t$, this can be done. It is clear that $K_{A_j}^{\{\mathfrak{A}_\alpha \mid \alpha \in I\}} = x_j$. Therefore, Γ is false in $\mathscr{P}^{\mathfrak{S}}\mathfrak{A}_\alpha (\alpha \in I)$, which contradicts the choice of Γ.

Problems and Complements

1. The associative law for direct products (Section 2.5) takes on the following form for filtered products.

Let $\{J_k \mid k \in K\}$ be a partition of the index set I, and let D_k and D^* be filters over J_k and K, respectively. Then the family

$$D = \{X \subseteq I \mid \{k \in K \mid X \cap J_k \in D_k\} \in D^*\}$$

is a filter over I and for any $\mathfrak{A}_\alpha$ $(\alpha \in I)$

$$\prod_{\alpha \in I} \mathfrak{A}_\alpha / D \cong \prod_{k \in K} (\prod_{\alpha \in J_k} \mathfrak{A}_\alpha / D_k) / D^*$$

(see Frayne, Morel, and Scott [15], p. 202).

2. If each $\mathfrak{A}_\alpha$ has cardinality $\geqq m$, then $|\prod \mathfrak{A}_\alpha / D| \geqq m$ for any filter $\langle I, D \rangle$ (m a natural number).

3. Let $\mathfrak{m}$ be an infinite cardinal and let $|\mathfrak{A}| \geqq \mathfrak{m} = |I|$. If the filter $\langle I, D \rangle$ is homogenous, then $|\mathfrak{A}^I / D| > \mathfrak{m}$ ([15], p. 206).

4. If the cardinal $\mathfrak{m}$ is infinite and $\mathfrak{n}$ is the least cardinal satsifying $\mathfrak{m} < 2^{\mathfrak{n}}$, then for any homogeneous filter $\langle I, D \rangle$, $|I| = \mathfrak{n}$ and $|\mathfrak{A}| \geqq \mathfrak{m}$ imply $|\mathfrak{A}^I / D| \geqq 2^{\mathfrak{n}}$ ([15], p. 206).

5. If $\langle I, D \rangle$ is an ultrafilter and $|\prod \mathfrak{A}_\alpha / D| < 2^{\aleph_0}$, then $\prod \mathfrak{A}_\alpha / D \cong \mathfrak{A}_i$ for some $i \in I$ ([15], p. 208).

6. If $\langle I, D \rangle$ is an arbitrary filter and $|\prod \mathfrak{A}_\alpha / D| < 2^{\aleph_0}$, then for some finite set $J \subseteq I$

$$\prod \mathfrak{A}_\alpha / D \cong \prod \mathfrak{A}_\beta \quad (\beta \in J)$$

([15], p. 210).

7. For any cardinal number $\aleph_\alpha$, a filter $\langle I, D \rangle$ is $\aleph_\alpha$-complete if for each $\mathfrak{m} < \aleph_\alpha$ the intersection of $\mathfrak{m}$ sets of D belongs to D. In particular, countably complete filters are $\aleph_1$-complete filters.

If $|\mathfrak{A}| < \mathfrak{m}$ and $\langle I, D \rangle$ is an $\mathfrak{m}$-complete ultrafilter, then $\mathfrak{A}^I / D \cong \mathfrak{A}$.

8. In Theorem 3 of Section 8, can we discard the requirement that the filter not be countably complete?

9. Which cardinals are the cardinalities of ultraproducts over nonprincipal ultrafilters? Does there exist a nonprincipal ultrafilter $\langle I, D \rangle$ for which

$$\aleph_0 < \aleph_0{}^I / D < 2^{|I|}?$$

([15] p. 208).

10. The classes of finite groups, finite abelian groups, free groups and cyclic groups are not axiomatizable.

9. Indistinguishability and Elementary Embedding

Suppose we are given some logical language Ξ and certain objects $\mathfrak{A}$ and $\mathfrak{B}$. We shall say that $\mathfrak{A}$ and $\mathfrak{B}$ are *indistinguishable in the language* Ξ or *Ξ-equivalent* if every property which can be formulated in the language Ξ and which holds for one object also holds for the other. Indistinguishability of objects depends on the power of the language. Objects indistinguishable in one language may be distinguishable in a richer language. The notion of indistinguishability of objects can also be extended to the elements of objects by considering only those properties which characterize how elements are situated within a given object and which can be formulated in the given language. The result of making this idea more precise gives rise to the important concept of *Ξ-embedding* of one object in another. If the objects are algebraic systems and the language Ξ is the first-order language, then we are led to the concepts of elementary indistinguishability of systems and elementary embedding of systems, whose basic features we investigate below.

9.1 Elementary Embedding. Algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ of fixed signature Ω are called *elementarily equivalent* or *indistinguishable* (symbolically, $\mathfrak{A} \equiv \mathfrak{B}$) if every closed first-order formula of signature Ω which is true in one of the given systems is also true in the other. From the abstractness of the first-order language (Section 6.3), it follows that any two isomorphic systems are indistinguishable. The converse is in general false. Take any infinite algebraic system $\mathfrak{A}$, and let I be an arbitrary set whose cardinality is equal to or greater than $|\mathfrak{A}|$. By Theorem 5 of Section 8.5, there exists an ultrafilter $\langle I, D_1 \rangle$, such that

$$|\mathfrak{A}^I/D_1| \geqq 2^{|I|} > |I|.$$

Thus, the systems $\mathfrak{A}$ and $\mathfrak{A}^I/D_1$ have different cardinality and hence are not isomorphic. At the same time they are indistinguishable. Indeed, if a closed formula $\mathscr{A}$ of signature Ω is true in $\mathfrak{A}$, then by the fundamental theorem on ultraproducts, $\mathscr{A}$ is also true in every ultrapower $\mathfrak{A}^I/D_1$. If $\mathscr{A}$ is false in $\mathfrak{A}$, then $\neg\mathscr{A}$ is true in $\mathfrak{A}$ and consequently, in $\mathfrak{A}^I/D_1$; that is, $\mathscr{A}$ is false in $\mathfrak{A}^I/D_1$. Thus, we have obtained the following result.

Theorem 1. *Every algebraic system $\mathfrak{A}$ is elementarily equivalent to any of its ultrapowers. If $\mathfrak{A}$ is infinite, there exist systems elementarily equivalent to $\mathfrak{A}$ but which are not isomorphic to $\mathfrak{A}$.*

It will be shown below that an algebraic system elementarily equivalent to a *finite* system $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$.

Recall that a first-order formula $\mathscr{A}$ is a formula of signature Ω with free variables $x_1, \ldots, x_n$ if all of its free variables are contained

in the set $\Omega \cup \{x_1, \ldots, x_n\}$. A formula $\mathscr{A}$ of signature Ω with free object variables $x_1, \ldots, x_n$ will be denoted by $\mathscr{A}(x_1, \ldots, x_n)$. The truth value of $\mathscr{A}(x_1, \ldots, x_n)$ in an algebraic system $\mathfrak{A}$ of signature Ω for given values $x_1 = a_1, \ldots, x_n = a_n$ $(a_i \in \mathfrak{A})$ will be denoted by $\mathscr{A}_{\mathfrak{A}}(a_1, \ldots, a_n)$ or simply by $\mathscr{A}(a_1, \ldots, a_n)$ if the algebraic system has been fixed beforehand.

Some of the considerations below hold not only for the full first-order language, but also for certain parts of it.

A collection of first-order formulas Υ will be called a *class* of formulas if in renaming free and bound object variables (see Section 6.2) in a formula of the class Υ (Υ-formula) we obtain a formula again in Υ.

An algebraic system $\mathfrak{A}$ of signature Ω is said to be Υ-*indistinguishable* from the algebraic system $\mathfrak{B}$ if $\mathfrak{B}$ has signature Ω and every closed Υ- formula of signature Ω which is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$. The notation $\mathfrak{A} \leqq_{\Upsilon} \mathfrak{B}$ will be used to indicate that $\mathfrak{A}$ is Υ-indistinguishable from $\mathfrak{B}$.

The collection of all closed Υ-formulas of signature Ω which are true in an algebraic system $\mathfrak{A}$ of signature Ω is called the Υ-*theory* for $\mathfrak{A}$ and will be denoted by $\Upsilon(\mathfrak{A})$. Therefore, if the algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ have the same signature, then $\mathfrak{A}$ is Υ-indistinguishable from $\mathfrak{B}$ if and only if $\Upsilon(\mathfrak{A}) \subseteq \Upsilon(\mathfrak{B})$.

Algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ are called Υ-*equivalent* (in symbols, $\mathfrak{A} \equiv_{\Upsilon} \mathfrak{B}$) if $\Upsilon(\mathfrak{A}) = \Upsilon(\mathfrak{B})$.

From the definitions, it is clear that for any class Υ, the relation $\leqq_{\Upsilon}$ is reflexive and transitive while the relation $\equiv_{\Upsilon}$ is reflexive, transitive and symmetric. Equally clear is that if $\Upsilon \subseteq \Upsilon^*$, then

$$\mathfrak{A} \leqq_{\Upsilon^*} \mathfrak{B} \Rightarrow \mathfrak{A} \leqq_{\Upsilon} \mathfrak{B}, \qquad \mathfrak{A} \equiv_{\Upsilon^*} \mathfrak{B} \Rightarrow \mathfrak{A} \equiv_{\Upsilon} \mathfrak{B}. \tag{1}$$

The collection of all first-order formulas will be denoted by E. Obviously, the notions of E-indistinguishability and E-equivalence are equivalent to the above notion of elementary equivalence of algebraic systems. From (1) it follows that Theorem 1 also holds for Υ-equivalence.

Clearly, if the class Υ satisfies the condition:

i) for every formula $\mathscr{A}$ in Υ, Υ also has a formula equivalent to $\neg\mathscr{A}$, then the relations $\leqq \Upsilon$ and $\equiv \Upsilon$ become equivalent. In particular, the relation of elementary indistinguishability becomes equivalent to the relation of elementary equivalence.

Consider algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ of some fixed signature Ω. A mapping $\sigma\colon \mathfrak{A} \to \mathfrak{B}$ is called an Υ-*embedding* of $\mathfrak{A}$ into $\mathfrak{B}$ if an arbitrary Υ-formula $\mathscr{A}(x_1, \ldots, x_n)$ with free object variables $x_1, \ldots, x_n$ and of signature Ω satisfies

$$\mathscr{A}(a_1, \ldots, a_n) \underset{\mathfrak{A}}{=} \mathsf{T} \Rightarrow \mathscr{A}(a_1\sigma, \ldots, a_n\sigma) \underset{\mathfrak{B}}{=} \mathsf{T}$$

for any $a_1, \ldots, a_n$ in $\mathfrak{A}$.

A submodel (in particular, a subsystem) $\mathfrak{A}$ of an algebraic system $\mathfrak{B}$ is called an Υ-*elementary submodel*, or briefly, an Υ-*submodel* in $\mathfrak{B}$ (in symbols, $\mathfrak{A} <_\Upsilon \mathfrak{B}$) if the identity mapping $a\sigma = a$ $(a \in \mathfrak{A})$ is an Υ-embedding of $\mathfrak{A}$ into $\mathfrak{B}$. A system $\mathfrak{A}$ is Υ-embeddable in $\mathfrak{B}$ if there exists an Υ-embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

Comparing the definitions of Υ-embeddability and Υ-indistinguishability, we immediately obtain the following two results.

Corollary 2. *Υ-embeddability of $\mathfrak{A}$ into $\mathfrak{B}$ implies that $\mathfrak{A}$ is Υ-indistinguishable from $\mathfrak{B}$. In particular,*

$$\mathfrak{A} <_\Upsilon \mathfrak{B} \Rightarrow \mathfrak{A} \leqq_\Upsilon \mathfrak{B}$$

for any Υ, $\mathfrak{A}$ and $\mathfrak{B}$.

Corollary 3. *If $\sigma\colon \mathfrak{A} \to \mathfrak{B}$ and $\varrho\colon \mathfrak{B} \to \mathfrak{C}$ are Υ-embeddings, then $\sigma\varrho\colon \mathfrak{A} \to \mathfrak{C}$ is also an Υ-embedding. In particular*

$$\mathfrak{A} <_\Upsilon \mathfrak{B}, \qquad \mathfrak{B} <_\Upsilon \mathfrak{C} \Rightarrow \mathfrak{A} <_\Upsilon \mathfrak{C}.$$

If a class of formulas Υ satisfies condition i) and $\sigma\varrho\colon \mathfrak{A} \to \mathfrak{C}$ and $\varrho\colon \mathfrak{B} \to \mathfrak{A}$ are Υ-embeddings then $\sigma\colon \mathfrak{A} \to \mathfrak{B}$ is also an Υ-embedding. In particular, assuming condition i), we have

$$\mathfrak{A} \subseteq \mathfrak{B}, \qquad \mathfrak{A} <_\Upsilon \mathfrak{C}, \qquad \mathfrak{B} <_\Upsilon \mathfrak{C} \Rightarrow \mathfrak{A} <_\Upsilon \mathfrak{B}$$

for any algebraic systems $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$.

We now formulate the simplest connection between Υ-embeddings and isomorphisms.

Corollary 4. *If the class Υ satisfies:*

ii) all atomic formulas and their negations are in Υ, then a mapping σ: $\mathfrak{A} \to \mathfrak{B}$ is an Υ-embedding of $\mathfrak{A}$ into $\mathfrak{B}$ if and only if σ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^\sigma$ and $\mathfrak{A}^\sigma <_\Upsilon \mathfrak{B}$.

The proof is evident.

We also point out that condition ii) implies that every Υ-submodel $\mathfrak{A}$ of the algebraic system $\mathfrak{B}$ is a subsystem of $\mathfrak{B}$.

E-embeddings and E-subsystems are called *elementary embeddings* and *elementary subsystems*. An important role in their theoretical development is played by so-called canonical embeddings of algebraic systems into filtered powers defined in the folloing way.

Let $\mathfrak{A}$ be an arbitrary algebraic system of signature Ω and let $\langle I, D\rangle$ be some filter. With each $a \in \mathfrak{A}$, we associate a constant function $t_a \in \mathfrak{A}^I$ defined by

$$t_a(\alpha) = a \qquad (\alpha \in I).$$

The mapping

$$\varepsilon\colon\ a \to t_a D$$

is called the *canonical embedding of* $\mathfrak{A}$ *into* $\mathfrak{A}^I/D$. If a and $b \in \mathfrak{A}$, $a \neq b$, then

$$\{\alpha \in I \mid t_a(\alpha) = t_b(\alpha)\} = \varnothing \notin D$$

and hence, $a\varepsilon \neq b\varepsilon$. In other words, ε is a one-to-one mapping of $\mathfrak{A}$ into $\mathfrak{A}^I/D$. Since for any n-ary predicate $P \in \Omega$ and any $a_1, \ldots, a_n$ in $\mathfrak{A}$,

$$P(t_{a_1} D, \ldots, t_{a_n} D) = \mathsf{T} \Leftrightarrow \{\alpha \in I \mid P(a_1, \ldots, a_n) = \mathsf{T}\} \in D,$$

it follows that

$$P(a_1, \ldots, a_n) = \mathsf{T} \Leftrightarrow P(a_1\varepsilon, \ldots, a_n\varepsilon) = \mathsf{T}.$$

In other words, *the canonical mapping* ε *is an isomorphism of* $\mathfrak{A}$ *onto the subsystem* $\mathfrak{A}^\varepsilon$ *of* $\mathfrak{A}^I/D$. Therefore the system $\mathfrak{A}$ is often identified with $\mathfrak{A}^\varepsilon$ (Section 2.3) and is viewed as a subsystem of $\mathfrak{A}^I/D$ whenever this does not result in ambiguity.

For which classes of formulas Υ and which filters D is the canonical mapping of $\mathfrak{A}$ into $\mathfrak{A}^I/D$ an Υ-embedding?

From Theorem 1 (Section 8.4) on the conditional filtration of Horn formulas, it follows that for any filter $\langle I, D\rangle$, the canonical embedding of $\mathfrak{A}$ into $\mathfrak{A}^I/D$ is an H-embedding, where H is the collection of all Horn formulas. However, of major import in what follows will be the following corollary, which is a direct consequence of the fundamental theorem on ultraproducts.

Corollary 5. *The canonical embedding of an algebraic system* $\mathfrak{A}$ *into any of its ultrapowers* $\mathfrak{A}^I/D$ *is an elementary embedding.*

If $\mathfrak{A} <_\Upsilon \mathfrak{B}$ for some class of formulas Υ, then $\mathfrak{B}$ is called an Υ-*extension* of $\mathfrak{A}$. E-extensions are called *elementary extensions.*

Corollary 6. *Any infinite algebraic system has elementary extensions whose cardinality is greater than any preassigned cardinal number* $\mathfrak{m}$.

Indeed, let the set I have cardinality $\mathfrak{m}$.

By Theorem 5 of Section 8.5, there is an ultrafilter D over I such that

$$|\mathfrak{A}^I/D| \geqq 2^{\mathfrak{m}} > \mathfrak{m}.$$

Identifying the system $\mathfrak{A}$ with its canonical image in $\mathfrak{A}^I/D$, we see that $\mathfrak{A}^I/D$ is an elementary extension of $\mathfrak{A}$ with the required cardinality.

For finite algebraic systems, the situation is different.

Corollary 7. *If the class of formulas* Υ *contains all formulas of the form*

$$(\forall x_1 \dots x_{n+1})\,(x_1 = x_2 \vee x_1 = x_3 \vee \dots \vee x_n = x_{n+1}), \tag{2}$$

then every finite Υ*-subsystem* $\mathfrak{A}$ *of an arbitrary algebraic system* $\mathfrak{B}$ *coincides with* $\mathfrak{B}$. *In particular, any ultrapower* $\mathfrak{A}^I/D$ *of a finite algebraic system* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{A}$.

Indeed, let n be the number of elements in $\mathfrak{A}$. Consider formula (2). By hypothesis, this formula belongs to Υ. It is true in $\mathfrak{A}$ and $\mathfrak{A} <_\Upsilon \mathfrak{B}$. Hence, (2) is also true in $\mathfrak{B}$, i.e. $\mathfrak{B}$ has at most n elements. Since $\mathfrak{A} \subseteq \mathfrak{B}$ and $|\mathfrak{B}| \leqq |\mathfrak{A}|$, we have $\mathfrak{A} = \mathfrak{B}$.

Let us introduce a useful notion. Consider a first-order formula $\mathscr{A}(x_1, \dots, x_m)$ with free object variables $x_1, \dots, x_m$. The formulas

$$(\exists x_1 \dots x_m)\,\mathscr{A}, \qquad (\forall x_1 \dots x_m)\,\mathscr{A}$$

will be called the $\exists$*-closure* and the $\forall$*-closure* (over the variables $x_1, \dots, x_m$) of formula $\mathscr{A}$.

The following more substantial theorem is from the paper of Frayne, Morel, and Scott [15].

Theorem 8. *Suppose that a class of formulas* Υ *satisfies the conditions*:

iii) for any two Υ*-formulas, there is a formula in* Υ *equivalent to their conjunction*;

iv) the $\exists$*-closure of any* Υ*-formula belongs to* Υ.

An algebraic system $\mathfrak{A}$ *is* Υ*-indistinguishable from the system* $\mathfrak{B}$ *if and only if* $\mathfrak{A}$ *is* Υ*-embeddable in some ultrapower* $\mathfrak{B}^I/D$ *of* $\mathfrak{B}$.

Sufficiency is obvious. Indeed if $\mathfrak{A} \leqq_\Upsilon \mathfrak{B}^I/D$, then by Theorem 1, $\mathfrak{B}^I/D \equiv_E \mathfrak{B}$ and hence, $\mathfrak{B}^I/D \leqq_\Upsilon \mathfrak{B}$. Applying Corollary 3, we obtain $\mathfrak{A} \leqq_\Upsilon \mathfrak{B}$.

Let us now prove necessity. With each element $a \in \mathfrak{A}$, we associate an object variable z_a, which we assume does not belong to the signature Ω of $\mathfrak{A}$ and $\mathfrak{B}$. Consider the collection I^* of all closed formulas of signature $\Omega \cup \{z_a \mid a \in \mathfrak{A}\}$. Every formula $\mathscr{A}$ in I^* is either a closed formula of signature Ω or a formula $\mathscr{A}(z_{a_1}, \dots, z_{a_n})$ of signature Ω containing the free object variables $z_{a_1}, \dots, z_{a_n}$ where $a_1, \dots, a_n$ are mutally distinct elements of $\mathfrak{A}$. Let I denote the set of formulas $\mathscr{A}(z_{a_1}, \dots, z_{a_n})$ in I^* for which $\mathscr{A}(a_1, \dots, a_n)$ is true in $\mathfrak{A}$. The set I is usually called the Υ*-description of system* $\mathfrak{A}$ and is denoted by $\mathscr{D}_\Upsilon(\mathfrak{A})$.

Suppose $\mathscr{A}(z_{a_1}, \dots, z_{a_n}) \in I$. By iv), the formula $(\exists z_{a_1} \dots z_{a_n})\,\mathscr{A}(z_{a_1}, \dots, z_{a_n})$ belongs to Υ, has signature Ω and is true in the system $\mathfrak{A}$. By hypothesis, $\mathfrak{A} \leqq_\Upsilon \mathfrak{B}$. Consequently, the formula $(\exists z_{a_1} \dots z_{a_n})\mathscr{A}(z_{a_1}, \dots, z_{a_n})$ is also true in $\mathfrak{B}$. That is, $\mathfrak{B}$ has elements $b_1, \dots, b_n$ such that

$$\mathscr{A}_{\mathfrak{B}}(b_1, \dots, b_n) = \mathsf{T}. \tag{3}$$

Suppose that to each formula $\mathscr{A} \in I = \mathscr{D}_\Upsilon(\mathfrak{A})$, the corresponding elements $b_1, \ldots, b_n$ in $\mathfrak{B}$ have somehow been chosen (for instance, using the Axiom of Choice). Define a function $f_{\mathscr{A}}(x)$ by

$$\begin{aligned} f_{\mathscr{A}}(a_i) &= b_i \quad (i = 1, \ldots, n), \\ f_{\mathscr{A}}(x) &= b \quad (x \in \mathfrak{A}, x \neq a_1, \ldots, x \neq a_n), \end{aligned} \tag{4}$$

where b is a fixed arbitrary element of $\mathfrak{B}$. Next, to each $a \in \mathfrak{A}$, define a function $g_a(t)$ from I into $\mathfrak{B}$, by the relation

$$g_a(\mathscr{A}) = f_{\mathscr{A}}(a) \qquad (\mathscr{A} \in I).$$

The correspondence $a \to g_a$ is a mapping of $\mathfrak{A}$ into $\mathfrak{B}^I$. However it need not satisfy the requirements of the theorem. To obtain an Υ-embedding, we must filter $\mathfrak{B}^I$ over a suitable filter. For every formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n}) \in I$, define

$$I_{\mathscr{A}} = \{\mathscr{P} \in I \mid \mathscr{A}(f_{\mathscr{P}}(a_1), \ldots, f_{\mathscr{P}}(a_n)) \underset{\mathfrak{B}}{=} \mathsf{T}\}. \tag{5}$$

Each set $I_{\mathscr{A}}$ is nonempty since (3) implies that $\mathscr{A} \in I_{\mathscr{A}}$. Moreover, any finite intersection $I_{\mathscr{B}} \cap \ldots \cap I_{\mathscr{C}}$ of sets of the form (5) is also nonempty. Indeed, by iii), the class Υ contains a formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ equivalent to $\mathscr{B} \,\&\, \ldots \,\&\, \mathscr{C}$. From (3)—(5) it is clear that $\mathscr{A} \in I_{\mathscr{B}} \cap \ldots \cap I_{\mathscr{C}}$.

Let D denote an ultrafilter over I containing all the sets $I_{\mathscr{A}}$ $(\mathscr{A} \in I)$. We shall show that the mapping

$$\sigma\colon\ a \to g_a D$$

is the required Υ-embedding of $\mathfrak{A}$ into $\mathfrak{B}^I/D$.

By definition, for each formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ in I we have

$$\mathscr{A}(a_1\sigma, \ldots, a_n\sigma) \underset{\mathfrak{B}^I/D}{=} \mathsf{T} \Leftrightarrow \{\mathscr{P} \in I \mid \mathscr{A}(g_{a_1}(\mathscr{P}), \ldots, g_{a_n}(\mathscr{P})) \underset{\mathfrak{B}}{=} \mathsf{T}\} \in D.$$

Also,

$$\begin{aligned} &\{\mathscr{P} \in I \mid \mathscr{A}(g_{a_1}(\mathscr{P}), \ldots, g_{a_n}(\mathscr{P})) \underset{\mathfrak{B}}{=} \mathsf{T}\} \\ &\qquad = \{\mathscr{P} \in I \mid \mathscr{A}(f_{\mathscr{P}}(a_1), \ldots, f_{\mathscr{P}}(a_n)) \underset{\mathfrak{B}}{=} \mathsf{T}\} \in D, \end{aligned}$$

and hence the expression $\mathscr{A}(a_1\sigma, \ldots, a_n\sigma)$ is true in $\mathfrak{B}^I/D$.

Thus, if an Υ-formula $\mathscr{A}(x_1, \ldots, x_n)$ of signature Ω with free object variables $x_1, \ldots, x_n$ is true in $\mathfrak{A}$ for $x_1 = a_1, \ldots, x_n = a_n$, then looking at the formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ in I, we see that $\mathscr{A}(a_1\sigma, \ldots, a_n\sigma)$ is true in $\mathfrak{B}^I/D$, as required.

We point out one corollary which follows immediately from Theorem 8.

Corollary 9. *Suppose that a class of formulas* Υ *contains the conjunctions of atomic formulas, the negations of atomic formulas and the* $\exists$*-closures of these conjunctions. Then, if an algebraic system* $\mathfrak{A}$ *is* Υ*-equivalent to a finite algebraic system* $\mathfrak{B}$*, then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are isomorphic.*

The class Υ satisfies the conditions of Theorem 8 and Corollary 4. Since $\mathfrak{A} \equiv_\Upsilon \mathfrak{B}$, by Theorem 8, there is an Υ-embedding σ of the system $\mathfrak{A}$ into some ultrapower $\mathfrak{B}^I/D$. By Corollary 4, σ is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}^I/D$. By hypothesis, the system $\mathfrak{B}$ is finite, and therefore $\mathfrak{B}^I/D$ is isomorphic to $\mathfrak{B}$. Thus, there is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, which implies $|\mathfrak{A}| \leqq |\mathfrak{B}|$.

Now exchanging the roles of $\mathfrak{A}$ and $\mathfrak{B}$, we obtain $|\mathfrak{A}| \geqq |\mathfrak{B}|$. Hence, $|\mathfrak{A}| = |\mathfrak{B}|$ and the above isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ is in fact an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

9.2 Elementary Subsystems. In the previous subsection we introduced the notion of an Υ-elementary subsystem of an algebraic system. We now wish to study in greater detail the properties of the collection of all Υ-subsystems of a given system. Throughout this subsection, we shall assume that the class Υ of first-order formulas satisfies the following:

i) All Υ-formulas are in prenex normal form. The renaming of free and bound variables in any Υ-formula yields another Υ-formula.

ii) To each Υ-formula $\mathscr{A}$, there is an Υ-formula equivalent to $\neg\mathscr{A}$.

iii) If $(Qx)\mathscr{A} \in \Upsilon$ $(Q = \exists, \forall)$, then $\mathscr{A} \in \Upsilon$.

Suppose we are given some subsystem $\mathfrak{A}$. As in Section 9.1, with each $a \in \mathfrak{A}$ we associate an object symbol z_a and we assume that the symbols z_a are not in the signature Ω of $\mathfrak{A}$. Let $\Omega_\mathfrak{A}$ denote the union of Ω and the collection of all symbols $z_a (a \in \mathfrak{A})$. The element a will be called the value of z_a. Let $\mathfrak{A}$ be a subsystem of $\mathfrak{B}$. Then $\Omega_\mathfrak{A} \subseteq \Omega_\mathfrak{B}$. In any closed formula $\mathscr{A}$ of signature $\Omega_\mathfrak{A}$, the only free object variables are the object symbols of Ω and the object symbols of the form z_a. Since both types of symbols take fixed values in $\mathfrak{A}$ and in $\mathfrak{B}$, the truth value of the formula $\mathscr{A}$ is well-defined in $\mathfrak{A}$ and in $\mathfrak{B}$.

A subsystem $\mathfrak{A}$ of an algebraic system $\mathfrak{B}$ of signature Ω is called an Υ-elementary subsystem (Υ-subsystem), in symbols $\mathfrak{A} <_\Upsilon \mathfrak{B}$, if every formula $\mathscr{A}$ of signature $\Omega_\mathfrak{A}$ which is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$.

As already shown in Section 9.1, property ii) implies that a subsystem $\mathfrak{A}$ of the algebraic system $\mathfrak{B}$ of signature Ω is Υ-elementary in $\mathfrak{B}$, if and only if each Υ-formula of signature $\Omega_\mathfrak{A}$ which is true in $\mathfrak{B}$ is also true in $\mathfrak{A}$.

A simple useful test for a subsystem to be elementary is contained in the following.

Theorem 1. *A subsystem $\mathfrak{A}$ of an algebraic system $\mathfrak{B}$ of signature Ω is Υ-elementary if and only if the fact that an Υ-formula of the form $(\exists x)\mathscr{F}(x)$ of signature $\Omega_{\mathfrak{A}}$ is true in $\mathfrak{B}$ implies that the formula $\mathscr{F}(z_a)$ is true in $\mathfrak{B}$ for some $a \in \mathfrak{A}$.*

The necessity part is evident. We shall prove sufficiency. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the conditions of Theorem 1. We must show that if a prenex Υ-formula $\mathscr{A}$ of signature Ω is true in $\mathfrak{B}$, then it is true in $\mathfrak{A}$. The proof will be by induction on the number n of quantifiers in $\mathscr{A}$. If $n = 0$, then the assertion is trivial. Suppose that the assertion is valid for Υ-formulas having $n \geqq 0$ quantifiers, and let $\mathscr{A}$ be of the form $(\exists x)\mathscr{B}(x)$, where $\mathscr{B}(x)$ has n quantifiers. By hypothesis, $(\exists x)\mathscr{B}(x)$ is true in $\mathfrak{B}$ and consequently, for some $a \in \mathfrak{A}$, the formula $\mathscr{B}(z_a)$ is true in $\mathfrak{B}$. But $\mathscr{B}(z_a)$ has n quantifiers and signature $\Omega_{\mathfrak{A}}$. By the induction assumption, $\mathscr{B}(z_a)$ is true in $\mathfrak{A}$, and hence $(\exists x)\mathscr{B}(x)$ is true in $\mathfrak{A}$.

If $\mathscr{A}$ is of the form $(\forall x)\mathscr{B}(x)$ and the formula $(\forall x)\mathscr{B}(x)$ is true in $\mathfrak{B}$, then $\mathscr{B}(z_a)$ is true in $\mathfrak{B}$ for any $a \in \mathfrak{A}$. By induction, $\mathscr{B}(z_a)$ being true in $\mathfrak{B}$ implies $\mathscr{B}(z_a)$ is true in $\mathfrak{A}$ and hence $(\forall x)\mathscr{B}(x)$ is true in $\mathfrak{A}$.

Theorem 1 leads directly to an "algebraic" proposition.

Corollary 2. *If to each finite collection $a_1, \ldots, a_n$ of elements in a subsystem $\mathfrak{A}$ of an algebraic system $\mathfrak{B}$ and to each $b \in \mathfrak{B}$, there exists an automorphism σ of $\mathfrak{B}$ leaving $a_1, \ldots, a_n$ fixed and mapping b into some element of $\mathfrak{A}$, then the subsystem $\mathfrak{A}$ is elementary in $\mathfrak{B}$.*

By Theorem 1, it is sufficient to prove that if a formula

$$\mathscr{A}(b, a_1, \ldots, a_n) \qquad (a_1, \ldots, a_n \in \mathfrak{A}, b \in \mathfrak{B}),$$

having only $a_1, \ldots, a_n, b$ as free symbols in the signature Ω, is true in $\mathfrak{B}$, then the formula $\mathscr{A}(a, a_1, \ldots, a_n)$ is true in $\mathfrak{B}$ for some $a \in \mathfrak{A}$. Since σ is an automorphism of $\mathfrak{B}$, we have (Section 6.3)

$$\mathscr{A}(b, a_1, \ldots, a_n) \underset{\mathfrak{B}}{=} \mathsf{T} \Leftrightarrow \mathscr{A}(b\sigma, a_1\sigma, \ldots, a_n\sigma) \underset{\mathfrak{B}}{=} \mathsf{T} \Leftrightarrow \mathscr{A}(b\sigma, a_1, \ldots, a_n) \underset{\mathfrak{B}}{=} \mathsf{T}$$

where $b\sigma \in \mathfrak{A}$, as required.

Theorem 3. (A. Tarski, R. Vaught [72]). *Any set M of elements of an algebraic system $\mathfrak{B}$ of signature Ω is contained in some elementary subsystem $\mathfrak{A}$ of $\mathfrak{B}$ such that the cardinality of $\mathfrak{A}$ does not exceed* $\max(|M|, |\Omega|, \aleph_0)$.

Let

$$\mathfrak{m} = \max(|M|, |\Omega|, \aleph_0)$$

and consider the collection S of all closed formulas of signature Ω_M of the form $(\exists x)\mathscr{A}(x)$ ($\mathscr{A}(x)$ may have quantifiers) which are true in $\mathfrak{B}$.

By Section 2.6, $|S| = \mathfrak{m}$. For each formula $(\exists x)\mathscr{A}(x) \in S$, choose some element a in $\mathfrak{B}$ such that $\mathscr{A}(a)$ is true in $\mathfrak{B}$. Let M_1 denote the collection of all such elements. The cardinality of M_1 does not exceed that of the set of all closed formulas of signature Ω_M, i.e. does not exceed $\mathfrak{m}$. Since for each formula of the form $(\exists x)(x = a)$ $(a \in M)$, we must choose just the element a, we have $M \subseteq M_1$.

Now we repeat the process, taking M_1 in place of M for our initial set of elements. The collection of all elements selected will be denoted by M_2. Continuing this process, we obtain a chain of sets

$$M \subseteq M_1 \subseteq M_2 \subseteq \ldots \tag{1}$$

Let A be the union of all the M_i and let $\mathfrak{A}$ be the submodel of the model $\mathfrak{B}$ with carrier A. From (1), it follows that

$$|\mathfrak{A}| \leqq \mathfrak{m} + \mathfrak{m} + \mathfrak{m} + \cdots = \aleph_0 \cdot \mathfrak{m} = \mathfrak{m}.$$

From Theorem 1, it is easy to see that the submodel $\mathfrak{A}$ is elementary in $\mathfrak{B}$. Indeed, let $(\exists x)\mathscr{A}(x)$ be some closed formula of signature $\Omega_{\mathfrak{A}}$ true in $\mathfrak{B}$. Since this formula contains only a finite number of symbols from $\Omega_{\mathfrak{A}}$, it is a formula of signature Ω_{M_i} for some sufficiently large i. Thus, in constructing the set M_{i+1}, we must have selected some element $a \in \mathfrak{B}$ for which $\mathscr{A}(a)$ is true in $\mathfrak{B}$, and put that element in M_{i+1}. Therefore, there exists an element $a \in \mathfrak{A}$ for which $\mathscr{A}(a)$ is true in $\mathfrak{B}$. By Theorem 1, we conclude that the model $\mathfrak{A}$ is elementary in $\mathfrak{B}$, and therefore, $\mathfrak{A}$ is not only a submodel of $\mathfrak{B}$ but also a subsystem satisfying the requirements of the theorem.

Corollary 4. *If the cardinality of an infinite algebraic system $\mathfrak{B}$ is at least that of its signature Ω, then to any set $M \subseteq \mathfrak{B}$ and any cardinal $\mathfrak{n}$ satisfying*

$$\max(|M|, |\Omega|, \aleph_0) \leqq \mathfrak{n} \leqq |\mathfrak{B}|, \tag{2}$$

there exists an elementary subsystem $\mathfrak{A}$ in $\mathfrak{B}$ of cardinality $\mathfrak{n}$ which contains M.

In particular, every uncountable algebraic system of finite or countable signature contains infinitely many countable elementary subsystems.

Indeed, let $M_1 = M \cup M_0$, where M_0 is an arbitrary set of elements of $\mathfrak{B}$ having cardinality $\mathfrak{n}$. From (2) it follows that $|M_1| = \mathfrak{n}$. According to Theorem 3, $\mathfrak{B}$ contains an elementary subsystem $\mathfrak{A}$ containing M_1 whose cardinality does not exceed $\max(|\Omega|, \aleph_0, |M_1|) = \mathfrak{n}$. Since $M_1 \subseteq \mathfrak{A}$, it follows that $|\mathfrak{A}| \geqq \mathfrak{n}$, and hence $|\mathfrak{A}| = \mathfrak{n}$.

By Section (9.1), an algebraic system $\mathfrak{B}$ is called an elementary extension of the system $\mathfrak{A}$ if $\mathfrak{A}$ is an elementary subsystem of $\mathfrak{B}$.

Theorem 5. *Every infinite algebraic system* $\mathfrak{A}$ *of signature* Ω *has an elementary extension* $\mathfrak{C}$ *of any desired cardinality* $\mathfrak{n}$ *greater than or equal to* $|\Omega| + |\mathfrak{A}|$.

Let I be an arbitrary set of cardinality $\mathfrak{n}$. By Theorem 5 of Section 8.5, there exists an ultrafilter $\langle I, D\rangle$ such that

$$|\mathfrak{A}^I/D| \geqq 2^{\mathfrak{n}} > \mathfrak{n}.$$

By Section 9.1, there is an elementary embedding of the algebraic system $\mathfrak{A}$ into the ultrapower $\mathfrak{A}^I/D$ and hence, there is an algebraic system $\mathfrak{B}$ isomorphic to $\mathfrak{A}^I/D$ containing $\mathfrak{A}$ as an elementary subsystem. Since $|\mathfrak{B}| > \mathfrak{n}$, there is an element $b \in \mathfrak{B}$ not in $\mathfrak{A}$. Putting $M = \mathfrak{A} \cup \{b\}$, we see that the given cardinal $\mathfrak{n}$ satisfies conditions (2) of Corollary 4. Hence there is an elementary subsystem $\mathfrak{C}$ in $\mathfrak{B}$ which contains the set M and has cardinality $\mathfrak{n}$. By Section 9.1 (see Corollary 3), we know

$$\mathfrak{A} \subseteq \mathfrak{C}, \quad \mathfrak{A} < \mathfrak{B}, \quad \mathfrak{C} < \mathfrak{B} \Rightarrow \mathfrak{A} < \mathfrak{C}.$$

Thus, $\mathfrak{C}$ satisfies all the conditions of Theorem 5.

It is easy to construct an example showing that it is not possible to replace the condition $\mathfrak{n} \geqq |\Omega| + |\mathfrak{A}|$ in Theorem 5 with the weaker condition $\mathfrak{n} \geqq |\mathfrak{A}|$. Let the signature Ω consist of the object symbols a_α and the unary operator symbols f_λ, where α is any finite sequence $\langle \alpha_1, \ldots, \alpha_m\rangle$, $\alpha_i \in \{0, 1\}$, $m = 1, 2, \ldots$, and λ is any infinite sequence $\langle \lambda_1, \lambda_2, \ldots\rangle$, $\lambda_i \in \{0, 1\}$. Also, let $[\lambda]_m = \langle \lambda_1, \ldots, \lambda_m\rangle$ and let A denote the collection of all finite sequences α. Setting $a_\alpha = \alpha$ and $f_\lambda(\alpha) = [\lambda]_m$, if the length of α is m, we convert A into an algebraic system $\mathfrak{A}$ of signature Ω. It is easy to see that in $\mathfrak{A}$ axioms of the form

$$x \neq a_\alpha \,\&\, \ldots \,\&\, x \neq a_\gamma \rightarrow f_\lambda(x) \neq f_\mu(x) \tag{3}$$

are true, where $a_\alpha, \ldots, a_\gamma$ are sequences of length less than m and $[\lambda]_m \neq [\mu]_m$. Therefore, the axioms (3) are true in every elementary extension $\mathfrak{B}$ of the system $\mathfrak{A}$. Suppose $\mathfrak{B} \neq \mathfrak{A}$ and $x \notin \mathfrak{A}$, $x \in \mathfrak{B}$. Then the antecedent of the implication in (3) is true for any $a_\alpha, \ldots, a_\gamma$ and hence, $f_\lambda(x) \neq f_\mu(x)$ for any $\lambda \neq \mu$. Thus, $\mathfrak{B}$ contains a continuum of distinct elements $f_\lambda(x)$ and so $|\mathfrak{B}| \geqq 2^{\aleph_0}$. Therefore $|\mathfrak{A}| = \aleph_0$ whereas every elementary extension of $\mathfrak{A}$ which does not coincide with $\mathfrak{A}$ is uncountable.

Corollary 4 and Theorem 5 lead to the following useful remark.

Corollary 6. *For every infinite algebraic system* $\mathfrak{A}$ *of arbitrary signature* Ω *and every infinite cardinal* $\mathfrak{m} \geqq |\Omega|$, *there exists an algebraic system* $\mathfrak{A}_1$ *of cardinality* $\mathfrak{m}$ *elementarily equivalent to* $\mathfrak{A}$.

Indeed, if $\mathfrak{m} \leqq |\mathfrak{A}|$, then by Corollary 4, there is an elementary subsystem $\mathfrak{A}_1$ of $\mathfrak{A}$ of cardinality $\mathfrak{m}$ which is elementarily equivalent to $\mathfrak{A}$. If $\mathfrak{m} > |\mathfrak{A}|$, then by Theorem 5, $\mathfrak{A}$ has an elementary extension $\mathfrak{A}_1$ of cardinality $\mathfrak{m}$ which is also elementarily equivalent to $\mathfrak{A}$, as required.

Corollary 4 guarantees that for uncountable systems of countable signature the collection of elementary subsystems is at least infinite. The following theorem gives the basic structural properties of this collection.

Theorem 7. *Let $\mathfrak{S}$ be a collection of Υ-subsystems $\mathfrak{A}_\alpha$ $(\alpha \in I)$ of an algebraic system $\mathfrak{B}$ of signature Ω. If $\mathfrak{S}$ is local in itself (see Section 2.3), then the union $\mathfrak{A}$ of the subsystems in $\mathfrak{S}$ is an Υ-subsystem of $\mathfrak{B}$.*

In particular, the union of an ascending chain of Υ-subsystems of $\mathfrak{B}$ is an Υ-subsystem of $\mathfrak{B}$.

We shall use Theorem 1. Suppose $(\exists x)\mathscr{A}(x)$ is an Υ-formula of signature $\Omega_{\mathfrak{A}}$ which is true in $\mathfrak{B}$. This formula involves only a finite number of symbols $z_a, \ldots, z_c$ $(a, \ldots, c \in \mathfrak{A})$. We must find an α such that the elements $a, \ldots, c$ are contained in $\mathfrak{A}_\alpha$. Since the formula $(\exists x)\mathscr{A}(x)$ has signature $\Omega_{\mathfrak{A}_\alpha}$, is true in $\mathfrak{B}$ and $\mathfrak{A}_\alpha <_\Upsilon \mathfrak{B}$, the given formula is also true in $\mathfrak{A}_\alpha$, i.e., $\mathscr{A}(d)$ is true in $\mathfrak{A}_\alpha$ for a suitable $d \in \mathfrak{A}_\alpha$. $\mathfrak{A}_\alpha <_\Upsilon \mathfrak{B}$ implies that $\mathscr{A}(d)$ is true in $\mathfrak{B}$. Since $d \in \mathfrak{A}$, by Theorem 1, $\mathfrak{A} <_\Upsilon \mathfrak{B}$.

10. Completeness and Model Completeness

Let us consider a class of structures (in the more general sense of the word) for which the notion of isomorphism has been defined. Suppose that a system of statements $\mathfrak{S}$ formulated in some language Ξ distinguishes a certain subclass from among the given structures. The system of statements (or axioms) $\mathfrak{S}$ is called *categorical* if all structures satisfying the properties $\mathfrak{S}$ are isomorphic. The system $\mathfrak{S}$ is called *complete* (in the language Ξ) if all the structures satisfying the properties $\mathfrak{S}$ are Ξ-equivalent, i.e. any two $\mathfrak{S}$-structures have exactly the same properties which admit formulation in the language Ξ (Ξ-equicharacteristic). If the language is a first-order language and the structures under consideration are algebraic systems of given signature, then the general notion of Ξ-completeness of a system of axioms becomes the precise concept of elementary completeness. This will be the topic studied in this section.

10.1. Complete Systems of Formulas. As in Section 9, let Υ denote class of first-order formulas invariant under change of name of the individual variables occurring in the formulas of this class.

Let $\mathfrak{S}$ be a collection of closed Υ-formulas of given signature Ω. We shall say that an algebraic system $\mathfrak{A}$ *satisfies* the set $\mathfrak{S}$, or that the

Indeed, if $\mathfrak{m} \leqq |\mathfrak{A}|$, then by Corollary 4, there is an elementary subsystem $\mathfrak{A}_1$ of $\mathfrak{A}$ of cardinality $\mathfrak{m}$ which is elementarily equivalent to $\mathfrak{A}$. If $\mathfrak{m} > |\mathfrak{A}|$, then by Theorem 5, $\mathfrak{A}$ has an elementary extension $\mathfrak{A}_1$ of cardinality $\mathfrak{m}$ which is also elementarily equivalent to $\mathfrak{A}$, as required.

Corollary 4 guarantees that for uncountable systems of countable signature the collection of elementary subsystems is at least infinite. The following theorem gives the basic structural properties of this collection.

Theorem 7. *Let $\mathfrak{S}$ be a collection of Υ-subsystems $\mathfrak{A}_\alpha$ $(\alpha \in I)$ of an algebraic system $\mathfrak{B}$ of signature Ω. If $\mathfrak{S}$ is local in itself (see Section 2.3), then the union $\mathfrak{A}$ of the subsystems in $\mathfrak{S}$ is an Υ-subsystem of $\mathfrak{B}$.*

In particular, the union of an ascending chain of Υ-subsystems of $\mathfrak{B}$ is an Υ-subsystem of $\mathfrak{B}$.

We shall use Theorem 1. Suppose $(\exists x)\mathscr{A}(x)$ is an Υ-formula of signature $\Omega_{\mathfrak{A}}$ which is true in $\mathfrak{B}$. This formula involves only a finite number of symbols $z_a, \ldots, z_c$ $(a, \ldots, c \in \mathfrak{A})$. We must find an α such that the elements $a, \ldots, c$ are contained in $\mathfrak{A}_\alpha$. Since the formula $(\exists x)\mathscr{A}(x)$ has signature $\Omega_{\mathfrak{A}_\alpha}$, is true in $\mathfrak{B}$ and $\mathfrak{A}_\alpha <_\Upsilon \mathfrak{B}$, the given formula is also true in $\mathfrak{A}_\alpha$, i.e., $\mathscr{A}(d)$ is true in $\mathfrak{A}_\alpha$ for a suitable $d \in \mathfrak{A}_\alpha$. $\mathfrak{A}_\alpha <_\Upsilon \mathfrak{B}$ implies that $\mathscr{A}(d)$ is true in $\mathfrak{B}$. Since $d \in \mathfrak{A}$, by Theorem 1, $\mathfrak{A} <_\Upsilon \mathfrak{B}$.

10. Completeness and Model Completeness

Let us consider a class of structures (in the more general sense of the word) for which the notion of isomorphism has been defined. Suppose that a system of statements $\mathfrak{S}$ formulated in some language Ξ distinguishes a certain subclass from among the given structures. The system of statements (or axioms) $\mathfrak{S}$ is called *categorical* if all structures satisfying the properties $\mathfrak{S}$ are isomorphic. The system $\mathfrak{S}$ is called *complete* (in the language Ξ) if all the structures satisfying the properties $\mathfrak{S}$ are Ξ-equivalent, i.e. any two $\mathfrak{S}$-structures have exactly the same properties which admit formulation in the language Ξ (Ξ-equicharacteristic). If the language is a first-order language and the structures under consideration are algebraic systems of given signature, then the general notion of Ξ-completeness of a system of axioms becomes the precise concept of elementary completeness. This will be the topic studied in this section.

10.1. Complete Systems of Formulas. As in Section 9, let Υ denote class of first-order formulas invariant under change of name of the individual variables occurring in the formulas of this class.

Let $\mathfrak{S}$ be a collection of closed Υ-formulas of given signature Ω. We shall say that an algebraic system $\mathfrak{A}$ *satisfies* the set $\mathfrak{S}$, or that the

set $\mathfrak{S}$ is *realized* in the system $\mathfrak{A}$, if the signature of $\mathfrak{A}$ is Ω and every formula of $\mathfrak{S}$ is true in $\mathfrak{A}$. The formulas $\mathfrak{S}$ are often called *axioms*, and the algebraic systems which realize the collection $\mathfrak{S}$ are called *models* of $\mathfrak{S}$. A set $\mathfrak{S}$ is called consistent (or satisfiable, cf. Section 6.4) if there exists at least one algebraic system satisfying $\mathfrak{S}$. A formula $\mathscr{A}$ is said to be a *consequence* of the collection $\mathfrak{S}$ if $\mathscr{A}$ holds in every algebraic system satisfying $\mathfrak{S}$.

A set of formulas $\mathfrak{S}$ is called Υ-*complete* in the signature Ω if $\mathfrak{S}$ is satisfiable and for every Υ-formula $\mathscr{A}$ of signature Ω, either $\mathscr{A}$ or $\neg\mathscr{A}$ is a consequence of the set $\mathfrak{S}$.

Let the symbol $\Upsilon\Omega$ denote the class of Υ-formulas which are of signature Ω. Obviously, the property of $\mathfrak{S}$ of being Υ-complete in the signature Ω is equivalent to being $\Upsilon\Omega$-complete. It is also clear that $\Upsilon\Omega$-completeness of $\mathfrak{S}$ implies $\Upsilon_1\Omega_1$-completeness for any $\Upsilon_1 \in \Upsilon$ and $\Omega_1 \subseteq \Omega$.

If Υ is an arbitrary set of first-order formulas, then Υ-completeness in the signature Ω is simply called *completeness in the signature* Ω (Ω-*completeness*). Finally, a set $\mathfrak{S}$ of formulas of fixed signature Ω is called *complete* if $\mathfrak{S}$ is complete in the signature Ω.

Suppose that the predicate symbol P is not contained in the signature Ω of a collection of formulas $\mathfrak{S}$. Then neither the formula $(\forall x)\,P(x, \ldots, x)$ nor its negation $(\exists x)\neg P(x, \ldots, x)$ is a consequence of $\mathfrak{S}$ and so $\mathfrak{S}$ can not be complete in any signature containing P. Similarly, if the object symbols a and b are not in Ω, then neither the formula $a = b$ nor the formula $a \neq b$ is a consequence of $\mathfrak{S}$ and hence, $\mathfrak{S}$ can not be complete in a signature containing the symbols a and b. Except for trivial cases, $\mathfrak{S}$ is also not complete in a signature containing an object symbol a which does not occur in any of the formulas of the collection $\mathfrak{S}$. Thus, when discussing the completeness of a collection of formulas $\mathfrak{S}$ of signature Ω complete in signature Ω_1, we may assume without loss of generality that $\Omega_1 \subseteq \Omega$.

The above definition of completeness can be put in another form.

Corollary 1. *A satisfiable collection of formulas* $\mathfrak{S}$ *is* Υ-*complete in the signature* Ω *if and only if all algebraic systems satisfying* $\mathfrak{S}$ *are* $\Upsilon\Omega$-*equivalent.*

Necessity. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are models for the collection $\mathfrak{S}$ that are not $\Upsilon\Omega$-equivalent. This means that there is an $\Upsilon\Omega$-formula $\mathscr{A}$ which is true in one of the models and not in the other. Therefore, neither $\mathscr{A}$, nor $\neg\mathscr{A}$ is a consequence of $\mathfrak{S}$.

Sufficiency. Suppose all algebraic systems satisfying $\mathfrak{S}$ are Υ-equivalent and $\mathscr{A}$ is any $\Upsilon\Omega$-formula. Then either $\mathscr{A}$ is true in all models of $\mathfrak{S}$ or $\neg\mathscr{A}$ is true in all models of $\mathfrak{S}$. In the first instance, $\mathscr{A}$ is a con-

sequence of $\mathfrak{S}$ and in the second, $\neg\mathscr{A}$ is a consequence of $\mathfrak{S}$. Thus, the collection $\mathfrak{S}$ is Υ-complete.

Corollary 2. (Lindenbaum's Theorem). *Every satisfiable collection of formulas $\mathfrak{S}$ of signature Ω can be extended to a complete collection of formulas of signature Ω.*

By hypothesis, the collection $\mathfrak{S}$ has some model $\mathfrak{A}$ of signature Ω. Let $T_\Omega(\mathfrak{A})$ be the collection of all formulas of signature Ω which are true in $\mathfrak{A}$. Clearly, $\mathfrak{S} \subseteq T_\Omega(\mathfrak{A})$. On the other hand, for every formula $\mathscr{A}$ of signature Ω, either $\mathscr{A}$ or $\neg\mathscr{A}$ is true in $\mathfrak{A}$. Hence, either $\mathscr{A}$ or $\neg\mathscr{A}$ is contained in $T_\Omega(\mathfrak{A})$ and so is a consequence of $T_\Omega(\mathfrak{A})$. Therefore, $T_\Omega(\mathfrak{A})$ is the desired extension of $\mathfrak{S}$.

A collection of formulas $\mathfrak{S}$ of fixed signature Ω is said to be *categorical in signature* $\Omega_1 \subseteq \Omega$ if $\mathfrak{S}$ is satisfiable and all its models of signature Ω are Ω_1-isomorphic. A collection $\mathfrak{S}$ is *categorical* if it is categorical in the signature Ω. Clearly, a collection of formulas categorical in the signature Ω is also categorical in every smaller signature. It is also clear that a collection $\mathfrak{S}$ categorical in the signature Ω_1 is Ω_1-complete. Do there exist any categorical collections of formulas?

Suppose that a collection of formulas $\mathfrak{S}$ of signature Ω has an infinite model $\mathfrak{A}$ of signature Ω. By Theorem 5 (Section 9.2), there is a model $\mathfrak{B}$ of signature Ω elementarily equivalent to $\mathfrak{A}$ whose cardinality is greater than that of $\mathfrak{A}$. The elementary equivalence of $\mathfrak{A}$ and $\mathfrak{B}$ implies that all the formulas of $\mathfrak{S}$ are true in $\mathfrak{B}$, i.e. $\mathfrak{B}$ is a model for $\mathfrak{S}$. At the same time, $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic since they have different cardinality. Therefore, the collection $\mathfrak{S}$ is not categorical. *No infinite algebraic system can be described to within isomorphism in a first-order language.* By contrast, every finite algebraic system is easily described to within isomorphism.

Let $\mathfrak{A}$ be any (finite or infinite) algebraic system of fixed signature Ω. With each element $a \in \mathfrak{A}$ we associate an object symbol z_a not belonging to Ω. As in Section 9.2, the symbol $\Omega_\mathfrak{A}$ will denote the union of Ω and the set of all symbols z_a $(a \in \mathfrak{A})$. The element a will be called the canonical value of z_a in the system $\mathfrak{A}$. The *diagram* of the system $\mathfrak{A}$ (denoted by $D(\mathfrak{A})$) is the collection of all atomic formulas and negations of atomic formulas of signature $\Omega_\mathfrak{A}$ which are true in $\mathfrak{A}$ for canonical values of the object symbols.

Comparing the definition of diagram of a system $\mathfrak{A}$ with the definition of its description $\mathscr{D}(\mathfrak{A})$ (Section 9.1), we see that the diagram is a subset of the description of a system. While the description of any system consists of an infinite number of formulas, the diagram of a *finite* algebraic system of *finite* signature consists of a finite number of formulas. For example, the diagram of the semigroup $\langle\{0, 1\}, \cdot\rangle$ (where

the dot is ordinary multiplication) is the following collection of formulas:

$$z_0 = z_0, \quad z_1 = z_1, \quad z_0 \neq z_1, \quad z_1 \neq z_1,$$

$$z_0 \cdot z_0 = z_0, \quad z_0 \cdot z_1 = z_0, \quad z_1 \cdot z_0 = z_0, \quad z_1 \cdot z_1 = z_1,$$

$$z_0 \cdot z_0 \neq z_1, \quad z_0 \cdot z_1 \neq z_1, \quad z_1 \cdot z_0 \neq z_1, \quad z_1 \cdot z_1 \neq z_0.$$

Consider an arbitrary satisfiable collection of formulas $\mathfrak{S}$ of fixed signature Ω. Let $\mathfrak{A}$ be a model for $\mathfrak{S}$. Adjoining to $\mathfrak{S}$ the diagram $D(\mathfrak{A})$, we obtain a collection of formulas $\mathfrak{S}_1 = \mathfrak{S} \cup D(\mathfrak{A})$ of signature $\Omega_{\mathfrak{A}}$. The formulas in $\mathfrak{S}$ are all true in $\mathfrak{A}$ for canonical values of the variables z_a.

Now consider any model $\mathfrak{B}$ of the collection $\mathfrak{S}_1$. Every variable z_a has a definite value b_a in $\mathfrak{B}$. Denote by σ the mapping $a \to b_a$ of the model $\mathfrak{A}$ into $\mathfrak{B}$. The discussion of Section 9.1 apropos of the descriptions of models shows that in this instance also, σ is an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Indentifying $\mathfrak{A}$ with the submodel $\mathfrak{A}^\sigma$, we see that every model of the collection $\mathfrak{S}_1$ contains the submodel $\mathfrak{A}$. The difference between a description and a diagram is that every model for $\mathfrak{S} \cup \mathscr{D}(\mathfrak{A})$ contains $\mathfrak{A}$ as an elementary submodel, but every model for $\mathfrak{S} \cup D(\mathfrak{A})$ contains $\mathfrak{A}$ simply as a submodel. These considerations lead in particular to the following statement.

Corollary 3. *Let $\mathfrak{A}$ be a finite algebraic system with n elements $a_1, \ldots, a_n$ and with signature Ω. Then all models of signature $\Omega_{\mathfrak{A}}$ for the collection of formulas*

$$D(\mathfrak{A}), \ (\forall x)\,(x = z_{a_1} \vee x = z_{a_2} \vee \ldots \vee x = z_{a_n})$$

are isomorphic to the model $\mathfrak{A}$. If the signature of the system $\mathfrak{A}$ is finite, then all models of signature Ω for the formula

$$(\exists z_{a_1} \ldots z_{a_n})\, D^*(\mathfrak{A}) \,\&\, (\forall x_1 \ldots, x_n x_{n+1})\,(x_1 = x_2 \vee x_1 = x_3 \vee \ldots \vee x_n = x_{n+1}),$$

where $D^*(\mathfrak{A})$ is the conjunction of the formulas in $D(\mathfrak{A})$, are isomorphic to $\mathfrak{A}$.

Thus, every finite system $\mathfrak{A}$ of infinite signature Ω can be described up to isomorphism by an infinite collection of formulas of signature $\Omega_{\mathfrak{A}}$, and every finite system $\mathfrak{A}$ of finite signature Ω can be described up to isomorphism by one formula of signature Ω.

Combining Corollary 1 with Corollary 7 of Section 9.1, we obtain the following.

Corollary 4. *Suppose that a collection of formulas $\mathfrak{S}$ of signature Ω is Υ-complete and Υ contains the $\exists$-closures of the conjunctions of atomic formulas and the negations of atomic formulas. If there exists a finite*

algebraic system $\mathfrak{A}$ *of signature* Ω *satisfying the collection* $\mathfrak{S}$, *then all models for* $\mathfrak{S}$ *are isomorphic to* $\mathfrak{A}$.

Corollaries 3 and 4 show that except for trivial cases, complete systems of formulas have nonisomorphic models of distinct cardinalities and are therefore not categorical. If one considers only those models of fixed infinite cardinality, then such models may turn out to be isomorphic. With this in mind, we introduce the following definition.

A collection of formulas $\mathfrak{S}$ of signature Ω is said to be *categorical in the cardinality* $\mathfrak{m}$ if $\mathfrak{S}$ has models of cardinality $\mathfrak{m}$ and all such models are isomorphic. Analogously, a collection of formulas $\mathfrak{S}$ is called $\Upsilon\Omega_1$-*complete in the cardinality* $\mathfrak{m}$ ($\Omega_1 \subseteq \Omega$) if $\mathfrak{S}$ has models of cardinality $\mathfrak{m}$ and all such models are $\Upsilon\Omega_1$-equivalent. Clearly, if $\mathfrak{S}$ is categorical in cardinality $\mathfrak{m}$, then $\mathfrak{S}$ is complete in this cardinality. The converse is not always true.

Theorem 5. (Vaught Test [73]). *If all models for the collection of formulas* $\mathfrak{S}$ *of signature* Ω *are infinite and the collection* $\mathfrak{S}$ *is* $\Upsilon\Omega_1$-*complete* ($\Omega_1 \subseteq \Omega$) *in some cardinality* $\mathfrak{m}$ *not less than* $|\Omega|$, *then* $\mathfrak{S}$ *is* $\Upsilon\Omega_1$-*complete.*

In particular, *if a collection of formulas* $\mathfrak{S}$ *of finite or countable signature has no finite models and is categorical in cardinality, then* $\mathfrak{S}$ *is complete.*

Suppose that $\mathfrak{S}$ is not $\Upsilon\Omega_1$-complete. Then it will have models $\mathfrak{A}$ and $\mathfrak{B}$ which are not $\Upsilon\Omega_1$-equivalent. This means that there is a closed $\Upsilon\Omega_1$-formula $\mathscr{A}$ true in one these models and not in the other. Suppose that $\mathscr{A}$ is true in $\mathfrak{A}$ and $\neg\mathscr{A}$ is true in $\mathfrak{B}$. By hypothesis, the model $\mathfrak{A}$ is infinite, the given cardinality $\mathfrak{m}$ is infinite and $\mathfrak{m} \geq |\Omega|$. By Corollary 6 of Section 9.2, this implies that there exists a model $\mathfrak{A}_1$ of cardinality $\mathfrak{m}$ elementarily equivalent to $\mathfrak{A}$. On the basis of the same corollary, we conclude that there exists a model $\mathfrak{B}_1$ of cardinality $\mathfrak{m}$ which is elementarily equivalent to $\mathfrak{B}$. From $\mathfrak{A} \equiv \mathfrak{A}_1$, it follows that the formula $\mathscr{A}$ is true in $\mathfrak{A}_1$. From $\mathfrak{B} \equiv \mathfrak{B}_1$, it follows that the formula $\neg\mathscr{A}$ is true in $\mathfrak{B}_1$. However, $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are models of cardinality $\mathfrak{m}$ and all the formulas of $\mathfrak{S}$ are true in each of them. By assumption $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are $\Upsilon\Omega_1$-equivalent; at the same time the $\Upsilon\Omega_1$-formula $\mathscr{A}$ is true in $\mathfrak{A}_1$ and false in $\mathfrak{B}_1$. The resulting contradiction proves the theorem.

Despite its simplicity and transparency, Vaught's theorem immediately yields completeness results for a number of collections of formulas that define very important algebraic concepts. Let us consider a few.

A (*linearly*) *ordered set* is defined by the axioms:

L 1. $(\forall xy)\,(x \leqq y \,\&\, y \leqq x \to x = y)$,

L 2. $(\forall xyz)\,(x \leqq y \,\&\, y \leqq z \to x \leqq z)$,

L 3. $(\forall xy)\,(x \leqq y \vee y \leqq x)$.

Adjoining the axiom

L 4. $(\forall xy)\,(\exists z)\,(x \leqq y \,\&\, x \neq y \to z \neq x \,\&\, z \neq y \,\&\, x \leqq z \,\&\, z \leqq y)$

we obtain the definition of the notion of a *dense (in itself) ordered set.* The axioms

L 5. $(\forall x)\,(\exists yz)\,(x \neq y \,\&\, x \neq z \,\&\, y \leqq x \,\&\, x \leqq z)$,
L 6. $(\exists x)\,(\forall y)\,(\exists z)\,(x \leqq y \,\&\, y \neq z \,\&\, y \leqq z)$,
L 7. $(\exists x)\,(\forall y)\,(\exists z)\,(y \leqq x \,\&\, y \neq z \,\&\, z \leqq y)$,
L 8. $(\exists xy)\,(\forall z)\,(x \neq y \,\&\, x \leqq y \,\&\, x \leqq z \,\&\, z \leqq y)$

define, respectively, ordered sets which are unbounded, bounded below and unbounded above, bounded above and unbounded below, bounded above and below. Hausdorff's Theorem (Section 1) asserts that all countable dense linearly ordered sets satisfying one of the requirements L5—L8 are isomorphic. In other words, Hausdorff's Theorem asserts that each of the collections of formulas {L1—L4, L5}, {L1—L4, L6}, {L1—L4, L7}, {L1—L4, L8} is complete in the cardinality $\aleph_0$. The signature of these collections consists only of the symbol $\leqq$ and all dense linearly ordered sets are infinite. According to Vaught's theorem, this means that each of these four collections of formulas is complete.

An even more interesting example is the collection of formulas that defines the notion of an algebraically closed field. For the signature of a field, we take the symbols $+$, $\cdot$, and the distinguished symbols 0 and 1. A *field* is defined by the axioms (see Section 4.1)

F 1. $0 \neq 1$,
F 2. $(\forall x)\,(0 + x = x \,\&\, 1 \cdot x = x)$,
F 3. $(\forall xyz)\,\big((x + y) + z = x + (y + z) \,\&\, x(yz) = (xy)z\big)$,
F 4. $(\forall xyz)\,\big(x(y + z) = xy + xz \,\&\, (y + z)x = yx + zx\big)$,
F 5. $(\forall xy)\,(x + y = y + x \,\&\, xy = yx)$,
F 6. $(\forall x)\,(\exists y)\,(x + y = 0)$,
F 7. $(\forall x)\,(\exists y)\,(x \neq 0 \to xy = 1)$.

The infinite sequence of axioms

F 8. $(\forall x_0 x_1 \ldots x_n)\,(\exists y)\,\big(x_0(x_0 y^n + x_1 y^{n-1} + \cdots + x_n) = 0\big)$
$(n = 2, 3, 4, \ldots)$

defines the concept of *algebraically closed field.*

A field is said to be of characteristic p (p a prime) if it satisfies the condition

F 9. $1 + 1 + \cdots + 1 = 0 \qquad (p = 2, 3, 5, \ldots)$,

where there are p terms on the left. Finally, a field has characteristic 0 if it satisfies the infinite system of axioms

$$\text{F}10.\ 1+1\neq 0,\ 1+1+1\neq 0,\ 1+1+1+1\neq 0,\ldots$$

By Steinitz's theorem (Section 4.2), algebraically closed fields are isomorphic if and only if they have the same characteristic and the same transcendence degree. If the transcendence degree is finite or countable, then the field is countable. If the transcendence degree is uncountable, then the cardinality of the field coincides with its transcendence degree. This shows that the collection of formulas {F1—F8} plus one of the formulas of F9 and the collection {F1—F8, F10} are categorical in any uncountable cardinality. Since in addition, all algebraically closed fields are infinite, by Vaught's theorem, we arrive at the conclusion that *the set of axioms for an algebraically closed field of fixed characteristic is complete.*

From this it follows, for instance, that the field of algebraic numbers and the field of all complex numbers are elementarily equivalent since they are both algebraically closed and have characteristic 0.

10.2 Model Completeness. Vaught's test for completeness reduces the question of completeness of a given collection of formulas $\mathfrak{S}$ to the question of whether the algebraic systems satisfying $\mathfrak{S}$ and having a certain fixed infinite cardinality are isomorphic. In many important cases, Vaught's test is inapplicable simply because the collection of formulas, though complete, is not categorical in any cardinality. Therefore, it is important to have other tests for completeness which do not depend on categoricity. One of the most powerful and most currently used tests is one based on the notion of model completeness introduced by A. Robinson and which is interesting from a purely theoretic standpoint.

In studying the properties of complete systems (Section 10.1), we found it useful to employ the diagram $D(\mathfrak{A})$ of an algebraic system $\mathfrak{A}$ and the notion of extended signature $\Omega_{\mathfrak{A}}$. Relying on these concepts, we introduce the following definition.

A collection of formulas $\mathfrak{S}$ of signature Ω is called *model complete* (in the given signature Ω) if for each algebraic system $\mathfrak{A}$ which satisfies the axioms $\mathfrak{S}$, the collection $\mathfrak{S} \cup D(\mathfrak{A})$ is complete in the signature $\Omega_{\mathfrak{A}}$.

For comparison, we introduce the auxiliary notion of weak completeness. A collection $\mathfrak{S}$ is *weakly complete* if for every system $\mathfrak{A}$ which satisfies $\mathfrak{S}$, the collection $\mathfrak{S} \cup D(\mathfrak{A})$ is complete in the original signature Ω.

It is clear that all complete and model complete collections are also

weakly complete. The converse is not true as is seen from the following simple examples.

Consider the collection $\mathfrak{S}$ of formulas F1—F8 of Section 10.1 that define the concept of algebraically closed field. All the formulas of F9 and F10 are independent of $\mathfrak{S}$, i.e., neither they nor their negations are consequences of $\mathfrak{S}$. Hence, the collection $\mathfrak{S}$ is not complete. At the same time, the diagram $D(\mathfrak{A})$ of any field $\mathfrak{A}$ contains either the formula $p \cdot 1 = 0$ (if the characteristic of $\mathfrak{A}$ is p) or the collection of formulas F10 (if the characteristic of $\mathfrak{A}$ is 0). Thus, the collection $\mathfrak{S} \cup D(\mathfrak{A})$ contains either the collection F1—F8, F10 which is complete in the signature $\Omega = \{+, \cdot, 0, 1\}$ or one of the collections consisting of F1—F8 and one of the formulas of F9, which are also complete in this signature. Therefore, for any field $\mathfrak{A}$, the collection $\mathfrak{S} \cup D(\mathfrak{A})$ is complete in the signature Ω. Thus, the collection $\mathfrak{S}$ is weakly complete. Moreover, with the help of a suitable modification of Vaught's test, one can prove that the formulas defining the concept of an algebraically closed field are not only weakly complete but also model complete.

To give an example of a complete but not model-complete system, let us examine the notion of a dense linearly ordered set bounded on the left and right. It is defined by formulas L1—L4, L8 of Section 10.1. Denote this collection by $\mathfrak{S}$ and add to it the diagram of the model $\mathfrak{A} = \langle [0, 1], \leqq \rangle$, where $[0, 1]$ is the set of all rational numbers x, $0 \leqq x \leqq 1$. The formula $(\forall x)(x \neq z_0 \to z_0 \leqq x)$ has signature $\Omega_{\mathfrak{A}}$, but neither it nor its negation is a consequence of $\mathfrak{S} \cup D(\mathfrak{A})$. Indeed, the above formula is true in the model $\langle [0, 1], \Omega_{\mathfrak{A}} \rangle$ but not in the model $\langle [-1, 1], \Omega_{\mathfrak{A}} \rangle$. Thus, the collection $\mathfrak{S}$ is complete but is not model complete.

An algebraic system $\mathfrak{A}_0$ is called a *prime system* or a *prime model* for a collection of formulas $\mathfrak{S}$ of signature Ω if $\mathfrak{A}_0$ satisfies $\mathfrak{S}$ and every system which satisfies $\mathfrak{S}$ contains a subsystem isomorphic to $\mathfrak{A}_0$.

Theorem 1. *If a weakly complete collection of formulas has a prime model, then the collection is complete.*

Let $\mathfrak{S}$ be a weakly complete collection of formulas of signature Ω and let $\mathfrak{A}_0$ be a prime model for $\mathfrak{S}$. By hypothesis, the collection $\mathfrak{S}_1 = \mathfrak{S} \cup D(\mathfrak{A}_0)$ is complete in the signature Ω. It is sufficient to prove that every formula $\mathscr{A}$ of signature Ω which follows from $\mathfrak{S}$, is true in any model $\mathfrak{B}$ of the collection $\mathfrak{S}$. The model $\mathfrak{B}$ contains a submodel $\mathfrak{B}_0$ isomorphic to $\mathfrak{A}_0$. Therefore, there is a model $\mathfrak{A}$ isomorphic to $\mathfrak{B}$ which has $\mathfrak{A}_0$ as a submodel. The formula $\mathscr{A}$ is a consequence of the collection $\mathfrak{S} \cup D(\mathfrak{A}_0)$, which is contained in the collection $\mathfrak{S} \cup D(\mathfrak{A})$. Therefore, $\mathscr{A}$ is a consequence of the latter. But $\mathfrak{B}$ is a model for this collection and hence $\mathscr{A}$ is true in $\mathfrak{A}$ and thus also in the model $\mathfrak{B}$ isomorphic to $\mathfrak{A}$.

In particular, it follows from Theorem 1 (A. Robinson [60]), that *every model-complete collection which has a prime model is complete.* However, there are complete systems which have prime models but are nevertheless not model complete. For example, every dense, linearly ordered set bounded on left and right contains a submodel isomorphic to the interval $\langle [0, 1], \leqq \rangle$ of rational numbers. Thus, this interval is a prime model for the collection of formulas L1—L4, L8 of Section 10.1 which defines the class of ordered sets with the indicated properties. We have seen that the above collection of formulas is complete, but not model complete.

In Section 10.1, it was shown that every satisfiable collection of formulas of fixed signature Ω can be extended to a complete collection of formulas of the same signature. The last example shows that for model-completeness this is false: there exist complete systems which are not model complete, i.e. systems which, because of completeness, cannot be further extended within the limits of the given signature. The situation changes if the signature can be enlarged.

Theorem 2. *The complete formula enlargement $\mathfrak{S}^*$ of any satisfiable collection of formulas $\mathfrak{S}$ is model-complete in the enlarged signature.*

The complete formula enlargement is constructed as follows. Let Ω be the signature of the given collection of formulas $\mathfrak{S}$. Consider all possible open formulas $\mathscr{A}_\lambda(x_1, \ldots, x_{n_\lambda})$ $(\lambda < \alpha)$ of signature Ω with free object variables $x_1, \ldots, x_{n_\lambda}$. For every such formula $\mathscr{A}_\lambda$, we introduce a special new n_λ-ary predicate symbol S_λ not in Ω and we write the formula

$$(\forall x_1 \ldots x_{n_\lambda})\left(S_\lambda(x_1, \ldots, x_{n_\lambda}) \leftrightarrow \mathscr{A}_\lambda(x_1, \ldots, x_{n_\lambda})\right). \tag{1}$$

Set $\Omega^* = \Omega \cup \{S_\lambda \mid \lambda < \alpha\}$ and denote by $\mathfrak{S}^*$ the union of the collection $\mathfrak{S}$ with the collection of all formulas (1). $\mathfrak{S}^*$ is the complete formula enlargement of $\mathfrak{S}$. We shall prove that the collection $\mathfrak{S}^*$ is model-complete in the signature Ω^*.

Let $\mathfrak{A}$ be an algebraic system of signature Ω^* which satisfies the formulas of $\mathfrak{S}$. Consider the collection

$$\mathfrak{S}^{**} = \mathfrak{S}^* \cup D(\mathfrak{A})$$

of signature $\Omega_{\mathfrak{A}}{}^*$. Suppose that we are given some formula $\mathscr{A}^*$ of signature $\Omega_{\mathfrak{A}}{}^*$. We may assume that $\mathscr{A}^*$ contains object symbols of the form $z_a (a \in \mathfrak{A})$ since otherwise we could then consider the equivalent formula $\mathscr{A}^* \,\&\, z_a = z_a$. Thus, let the formula $\mathscr{A}^*$ have in the signature Ω^* the free object variables $z_{a_1}, \ldots, z_{a_n}$. Replacing them by new object symbols $y_1, \ldots, y_n$ which do not occur in $\mathscr{A}^*$ or in $\Omega_{\mathfrak{A}}{}^*$, we obtain the formula $\mathscr{A}^*(y_1, \ldots, y_n)$. This formula has signature Ω^* and hence may contain

subformulas of the form $S_\lambda(a_1, \ldots, a_{n_\lambda})$ where each a_i is either one of the symbols $y_1, \ldots, y_n$, a signature symbol or a bound object variable of the formula $\mathscr{A}^*$. Corresponding to the symbol S_λ is a definite formula $\mathscr{A}_\lambda(x_1, \ldots, x_{n_\lambda})$ of signature Ω such that the collection $\mathfrak{S}^*$ contains formula (1). Replacing in $\mathscr{A}^*(y_1, \ldots, y_n)$ every atomic subformula of the form $S_\lambda(a_1, \ldots, a_{n_\lambda})$ by the corresponding formula $\mathscr{A}_\lambda(a_1, \ldots, a_{n_\lambda})$, we obtain some formula $\mathscr{A}(y_1, \ldots, y_n)$ of signature Ω. Since the collection $\mathfrak{S}^*$ contains all formulas of the form (1), $\mathfrak{S}^*$ implies the formula

$$(\forall y_1 \ldots y_n)\left(\mathscr{A}^*(y_1, \ldots, y_n) \leftrightarrow \mathscr{A}(y_1, \ldots, y_n)\right). \tag{2}$$

According to the construction, formula $\mathscr{A}(x_1, \ldots, x_n)$ has in Ω^* a corresponding predicate symbol S_β such that the collection $\mathfrak{S}^*$ contains the formula

$$(\forall x_1 \ldots x_n)\left(S_\beta(x_1, \ldots, x_n) \leftrightarrow \mathscr{A}(x_1, \ldots, x_n)\right). \tag{3}$$

Since formulas (2) and (3) follow from $\mathfrak{S}^*$, the formula

$$(\forall y_1 \ldots y_n)\left(S_\beta(y_1, \ldots, y_n) \leftrightarrow \mathscr{A}^*(y_1, \ldots, y_n)\right.$$

is also a consequence of $\mathfrak{S}^*$. This in turn implies the formulas

$$S_\beta(z_{a_1}, \ldots, z_{a_n}) \rightarrow \mathscr{A}^*(z_{a_1}, \ldots, z_{a_n}), \tag{4}$$

$$\neg S_\beta(z_{a_1}, \ldots, z_{a_n}) \rightarrow \neg \mathscr{A}^*(z_{a_1}, \ldots, z_{a_n}). \tag{5}$$

The symbol S_β occurs in the signature of the system $\mathfrak{A}$. Therefore, either the formula $S_\beta(z_{a_1}, \ldots, z_{a_n})$ or $\neg S_\beta(z_{a_1}, \ldots, z_{a_n})$ belongs to $D(\mathfrak{A})$. In the first case, $\mathfrak{S}^{**}$ has by (4) and (5) formula $\mathscr{A}^*(z_{a_1}, \ldots, z_{a_n})$ as a consequence. In the second case, $\neg \mathscr{A}^*(z_{a_1}, \ldots, z_{a_n})$ is a consequence of $\mathfrak{S}^{**}$. Therefore, the collection $\mathfrak{S}^{**}$ is complete in the signature $\Omega_{\mathfrak{A}}{}^*$, and Theorem 2 is proved.

Recall that to each collection of formulas $\mathfrak{S}$ of signature Ω, the *$\mathfrak{S}$-systems* (*$\mathfrak{S}$-subsystems*) are algebraic systems whose signature includes Ω and in which all the formulas of $\mathfrak{S}$ are satisfiable. The following property of model-complete collections is sometimes taken as the definition of model completeness.

Theorem 3. *A satisfiable collection of formulas $\mathfrak{S}$ of signature Ω is model complete if and only if all the $\mathfrak{S}$-subsystems of any $\mathfrak{S}$-system are Ω-elementary.*

Sufficiency. Take any $\mathfrak{S}$-system $\mathfrak{A}$ and set

$$\mathfrak{S}^* = \mathfrak{S} \cup D(\mathfrak{A}). \tag{6}$$

We must show that every formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ $(a_i \in \mathfrak{A})$ of signature Ω which is true in $\mathfrak{A}$ is a consequence of the collection $\mathfrak{S}^*$, i.e. that every such formula is true in any model $\mathfrak{B}$ for $\mathfrak{S}^*$. Since $\mathfrak{S}^*$ contains the diagram of $\mathfrak{A}$, system $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{B}$. By hypothesis, this implies that $\mathfrak{A}$ is an Ω-elementary subsystem of $\mathfrak{B}$ and hence, the formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ which is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$.

Necessity. Let $\mathfrak{B}$ be some $\mathfrak{S}$-system and $\mathfrak{A}$ an arbitrary $\mathfrak{S}$-subsystem of it. Let $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ $(a_i \in \mathfrak{A})$ be a formula with free variables $z_{a_1}, \ldots, z_{a_n}$ and signature Ω. By assumption, the collection $\mathfrak{S}^*$ defined by (6) is complete in the signature $\Omega_{\mathfrak{A}}$. If $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ is a consequence of $\mathfrak{S}^*$, then $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ is true in $\mathfrak{A}$ and in $\mathfrak{B}$. If $\neg\, \mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ is a consequence of $\mathfrak{S}^*$, then $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ is false in $\mathfrak{A}$ and in $\mathfrak{B}$. Thus, every formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ of signature Ω which is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$ and so the subsystem $\mathfrak{A}$ is Ω-elementary in $\mathfrak{B}$.

Theorem 3 gives a sufficient condition for model completeness. However, it is a very complex condition since its verification requires the examination of all formulas of the form $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ for any $\mathfrak{S}$-system $\mathfrak{A}$ and any $\mathfrak{S}$-subsystem $\mathfrak{B}$. In fact, instead of arbitrary formulas $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$, it is sufficient to consider only formulas of the form

$$(\exists x_1 \ldots x_m)\, \mathscr{B}(x_1, \ldots, x_m, z_{a_1}, \ldots, z_{a_n}), \tag{7}$$

where $\mathscr{B}$ is a conjunction of atomic formulas and negations of atomic formulas. Formulas of the form (7) are called *primitive.*

In accordance with Section 9.1, a subsystem $\mathfrak{A}$ of system $\mathfrak{B}$ will be called *primitive* (or *primitively elementary*) in $\mathfrak{B}$ if the truth of any primitive formula of signature $\Omega_{\mathfrak{A}}$ (i.e. formulas of the form (7)) in $\mathfrak{B}$ implies its truth in $\mathfrak{A}$.

Theorem 4 (Model Completeness Criterion of A. Robinson [60]). *A satisfiable collection of formulas $\mathfrak{S}$ of signature Ω is model complete if and only if all $\mathfrak{S}$-subsystems of any $\mathfrak{S}$-system $\mathfrak{B}$ are primitive subsystems of $\mathfrak{B}$.*

Since a subsystem which is elementary in a system is also primitive in that system, the necessity of the conditions of Theorem 4 follows from Theorem 3. We shall prove sufficiency.

For the moment we shall say that a formula $\mathscr{A}(x_1, \ldots, x_n)$ of signature Ω with free object variables $x_1, \ldots, x_n$ is $\mathfrak{S}$-strong if for any $\mathfrak{S}$-system $\mathfrak{A}$ and all $a_1, \ldots, a_n$ in $\mathfrak{A}$, either the formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ or the formula $\neg\, \mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ is a consequence of the collection $\mathfrak{S} \cup D(\mathfrak{A})$. We must show that there are no formulas of signature Ω which are not strong. Suppose, to the contrary, that there are formulas of the above type which are not strong. Since formulas equivalent to

non-strong formulas are themselves not strong, there exist prenex formulas which are not strong. Let $\mathscr{A}(x_1, \ldots, x_n)$ be a prenex formula with a minimal number k of quantifiers and which is not strong. Since all quantifier-free formulas of signature Ω are obviously strong, we must have $k \geqq 1$. From the definition it follows that $\neg \mathscr{A}$ is not strong whenever $\mathscr{A}$ is not strong. Of the formulas $\mathscr{A}$ and $\neg \mathscr{A}$, let $(\exists y)\mathscr{B}(y, x_1, \ldots, x_n)$ be the one whose prenex normal form begins with an existential quantifier. Thus, the formula $(\exists y)\mathscr{B}(y, x_1, \ldots, x_n)$ is not strong, but the formula $\mathscr{B}(x_0, x_1, \ldots, x_n)$ is strong. The former means that for a suitable $\mathfrak{S}$-system $\mathfrak{A}$ and suitable elements $a_1, \ldots, a_n \in \mathfrak{A}$ each of the collections

$$\mathfrak{S} \cup D(\mathfrak{A}) \cup \{(\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n})\} \tag{8}$$

and

$$\mathfrak{S} \cup D(\mathfrak{A}) \cup \{\neg (\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n})\} \tag{9}$$

is satisfiable.

Consider an arbitrary system $\mathfrak{B}$ which satisfies the axioms (8). Since collection (8) contains the formula $(\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n})$, there is an element $b \in \mathfrak{B}$ such that the formula $\mathscr{B}(z_b, z_{a_1}, \ldots, z_{a_n})$ is true in $\mathfrak{B}$. But the formula $\mathscr{B}(x_0, x_1, \ldots, x_n)$ is $\mathfrak{S}$-strong, $\mathfrak{B}$ is a $\mathfrak{S}$-system and for the element $b, a_1, \ldots, a_n$ in $\mathfrak{B}$, the formula $\mathscr{B}(z_b, z_{a_1}, \ldots, z_{a_n})$ is true in $\mathfrak{B}$. Therefore, the formula $\mathscr{B}(z_b, z_{a_1}, \ldots, z_{a_n})$ is a consequence of the collection

$$\mathfrak{S} \cup D(\mathfrak{B}). \tag{10}$$

The formula $\mathscr{B}(z_b, z_{a_1}, \ldots, z_{a_n})$ formally implies the formula $(\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n})$, and so the latter is a consequence of collection (10).

The collection (10) is, in general, infinite, but by the compactness theorem, there is a finite subset M of the collection (10) which also has the formula $(\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n})$ as a consequence. Denoting by $\mathfrak{S}_1$ the conjunction of the formulas of $\mathfrak{S}$ which are in M and by D_1 the conjunction of the formulas in $D(\mathfrak{B})$ which are in M, we see that the formula

$$\mathfrak{S}_1 \,\&\, D_1 \to (\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n}) \tag{11}$$

is identically true.

The formula $\mathfrak{S}_1$ is closed in the signature Ω, but the formula D_1 may contain free object variables. These will be symbols of the form $z_c (c \in \mathfrak{B})$. Some will coincide with the symbols $z_{a_1}, \ldots, z_{a_n}$ and some will be new symbols of the form $z_{c_1}, \ldots, z_{c_p}$ where $a_1, \ldots, a_n, c_1, \ldots, c_p$ are pairwise distinct elements of the system $\mathfrak{B}$. Therefore, formula (11) can be rewritten in the form

$$\mathfrak{S}_1 \,\&\, D_1(z_{a_1}, \ldots, z_{a_n}, z_{c_1}, \ldots, z_{c_p}) \to (\exists y)\mathscr{B}(y, z_{a_1}, \ldots, z_{a_n}). \tag{12}$$

Since D_1 is a subset of the diagram of $\mathfrak{B}$, it is clear that the primitive formula

$$(\exists z_{c_1} \ldots . z_{c_p}) D_1 (z_{a_1}, \ldots, z_{a_n}, z_{c_1}, \ldots, z_{c_p})$$

is true in $\mathfrak{B}$. By the assumption of the theorem, this implies that there are elements $a_{n+1}, \ldots, a_{n+p}$ in $\mathfrak{A}$ for which the formula

$$D_1 (z_{a_1}, \ldots, z_{a_n}, z_{a_{n+1}}, \ldots, z_{a_{n+p}}) \tag{13}$$

is true.

In the identically true formula (12), we replace the free object variables $z_{c_1}, \ldots, z_{c_p}$ by the corresponding variables $z_{a_{n+1}}, \ldots, z_{a_{n+p}}$ obtaining the formula

$$\mathfrak{S}_1 \,\&\, D_1 (z_{a_1}, \ldots, z_{a_n}, z_{a_{n+1}}, \ldots, z_{a_{n+p}}) \to (\exists y)\, \mathscr{B}(y, z_{a_1}, \ldots, z_{a_n}), \tag{14}$$

which is also identically true. Formula (13) is a conjunction of a subset of the formulas in $D(\mathfrak{A})$. This together with formula (14) means that the formula

$$(\exists y)\, \mathscr{B}(y, z_{a_1}, \ldots, z_{a_n})$$

is a consequence of the collection

$$\mathfrak{S} \cup D(\mathfrak{A}),$$

which contradicts the satisfiability of collection (9).

Suppose Υ is a class of formulas of a certain form. We shall say that a class $\mathfrak{K}$ of systems of signature Ω is Υ-*reducible*, if to each formula $\mathscr{A}(x_1, \ldots, x_n)$ of signature Ω with free object variables $x_1, \ldots, x_n$ there is a formula $\mathscr{B}(x_1, \ldots, x_n)$ of the form Υ of signature Ω such that the formula

$$(\forall x_1 \ldots . x_n) \left(\mathscr{A}(x_1, \ldots, x_n) \leftrightarrow \mathscr{B}(x_1, \ldots, x_n)\right) \tag{15}$$

is true in every $\mathfrak{K}$-system. In other words, for the class $\mathfrak{K}$ every formula predicate is equivalent to a predicate defined by a suitable formula of the form Υ.

In particular, a class $\mathfrak{K}$ of systems is called $\exists$-reducible, if every formula $\mathscr{A}(x_1, \ldots, x_n)$ is equivalent, for this class, to a suitable $\exists$-formula. It is clear that if a class $\mathfrak{K}$ is $\exists$-reducible, then it is also $\forall$-reducible since the equivalence of $\neg\, \mathscr{A}$ to some $\exists$-formula $\mathscr{B}$ implies the equivalence of $\mathscr{A}$ to the $\forall$-formula $\neg\, \mathscr{B}$.

A collection $\mathfrak{S}$ of closed formulas of signature Ω is called Υ-*reducible* if the class of all systems of signature Ω in which all the formulas of $\mathfrak{S}$ are true is Υ-reducible, that is, if the formulas (15) are consequences of the collection $\mathfrak{S}$.

Theorem 5 (A. Robinson [60]). *A collection of formulas is model complete if and only if it is* $\exists$*-reducible.*

Sufficiency. To this end, we use Theorem 3. Suppose the collection $\mathfrak{S}$ is $\forall$-reducible and $\mathfrak{A}$ is some $\mathfrak{S}$-subsystem of a $\mathfrak{S}$-system $\mathfrak{B}$. Then each formula $\mathscr{A}(z_{a_1}, \ldots, z_{a_n})$ will be equivalent in $\mathfrak{B}$ to a suitable $\forall$-formula

$$(\forall y_1 \ldots y_s)\, \mathscr{B}(y_1, \ldots, y_s, z_{a_1}, \ldots, z_{a_n}). \tag{16}$$

The truth of formula (16) in any system implies its truth in each subsystem containing the elements $a_1, \ldots, a_n$. Therefore, if $\mathscr{A}$ is true in $\mathfrak{B}$, then $\mathscr{A}$ is also true in $\mathfrak{A}$. By Theorem 3, this shows that $\mathfrak{S}$ is a model complete collection.

Necessity. Suppose that the collection of formulas $\mathfrak{S}$ is model complete, and suppose we are given some formula $\mathscr{A}(x_1, \ldots, x_n)$ of signature Ω with free object variables $x_1, \ldots, x_n$. We enlarge the signature Ω by adding to it new object symbols $a_1, \ldots, a_n$ and set $\Omega_1 = \Omega \cup \{a_1, \ldots, a_n\}$.

Denote by $\mathfrak{K}$ the class of systems of signature Ω_1 defined by the collection of formulas $\mathfrak{S}$. Let $\mathfrak{L}$ be the subclass of those $\mathfrak{K}$-systems in which the formula $\mathscr{A}(a_1, \ldots, a_n)$ is true. The subclass $\mathfrak{L}$ is hereditary in $\mathfrak{K}$. Indeed, let $\mathfrak{A}$ be any $\mathfrak{K}$-subsystem of $\mathfrak{L}$-system $\mathfrak{B}$. Both $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the formulas of $\mathfrak{S}$. Therefore, $\mathfrak{A}$ is an elementary subsystem in $\mathfrak{B}$. By hypothesis, the formula $\mathscr{A}(a_1, \ldots, a_n)$ $(a_1, \ldots, a_n \in \mathfrak{A})$ is true in $\mathfrak{B}$. Since $\mathfrak{A}$ is elementary in $\mathfrak{B}$, it follows that this formula is also true in $\mathfrak{A}$, i.e. $\mathfrak{A} \in \mathfrak{L}$.

Since the classes $\mathfrak{K}$ and $\mathfrak{L}$ are axiomatizable, they are ultraclosed. Since $\mathfrak{K}$ and $\mathfrak{L}$ are ultraclosed and $\mathfrak{L}$ is hereditary in $\mathfrak{K}$, it follows that the subclass $\mathfrak{L}$ is universally axiomatizable in $\mathfrak{K}$ (Section 7.2). But within $\mathfrak{K}$, $\mathfrak{L}$ is axiomatized by the single axiom $\mathscr{A}(a_1, \ldots, a_n)$. Therefore, $\mathfrak{L}$ is axiomatizable within $\mathfrak{K}$ by a single $\forall$-formula. Let it be the formula

$$(\forall y_1 \ldots y_s)\, \mathscr{B}(y_1, \ldots, y_s, a_1, \ldots, a_n). \tag{17}$$

The subclass $\mathfrak{L}$ is distinguished in the class $\mathfrak{K}$ by any of the formulas (17) or $\mathscr{A}(a_1, \ldots, a_n)$. Hence, in $\mathfrak{K}$

$$\mathscr{A}(a_1, \ldots, a_n) \leftrightarrow (\forall y_1 \ldots y_s)\, \mathscr{B}(y_1, \ldots, y_s, a_1, \ldots, a_n)$$

and therefore $\mathfrak{S}$ implies the equivalence

$$(\forall x_1 \ldots x_n)\left(\mathscr{A}(x_1, \ldots, x_n) \leftrightarrow (\forall y_1 \ldots y_s)\, \mathscr{B}(y_1, \ldots, y_s, x_1, \ldots, x_n)\right),$$

thus showing that the collection of formulas $\mathfrak{S}$ is $\forall$-reducible.

Remembering that a collection of formulas $\mathfrak{S}$ is $\forall$-complete in the signature Ω if for each closed $\forall$-formula Φ of signature Ω either Φ or $\neg\Phi$ follows from $\mathfrak{S}$, we obtain the following.

Corollary 6. *In order that a model-complete collection of formulas* $\mathfrak{S}$ *be complete, it is sufficient* (*and obviously necessary*) *that* $\mathfrak{S}$ *be* $\forall$*-complete.*

Indeed, by Theorem 5 the collection $\mathfrak{S}$ is $\forall$-reducible and so any closed formula $\mathscr{A}$ of signature Ω is equivalent in the class $K\mathfrak{S}$ (see Section 6.4) to a suitable closed $\forall$-formula of signature Ω.

Problems and Complements

1. If a collection $\mathfrak{S}$ of closed formulas of finite or countable signature is categorical in an uncountable cardinality, then it is categorical in any other uncountable cardinality (Morley [51]).

2. Give an example of a collection of closed formulas categorical in the cardinality of the continuum but not categorical in a countable cardinality.

Chapter V

Quasivarieties

Varieties and quasivarieties of algebras are classes of algebras which can be defined by means of a collection of identities and quasi-identities (conditional identities), respectively. Since identities and quasi-identities are $\forall$-formulas, varieties and quasivarieties are particular types of universally axiomatizable classes of algebras whose general properties were studied in Section 7. The special position of quasivarieties and especially varieties in general algebra is due to the fact that many structures intensely investigated in classical algebra such as groups, rings, lattices, Boolean algebras, etc., form varieties.

Questions of embeddability of systems of one variety into systems of another variety, the theory of defining relations and certain other aspects of algebra find their most natural expression in the theory of quasivarieties.

Throughout what follows, when speaking of classes of algebraic systems we shall mean only *abstract* classes (see Section 8.2)

11. General Properties

11.1 Characteristic Properties. Consider the algebraic systems of some fixed signature Ω. In accordance with Section 7.1, formulas of the form

$$P(\mathfrak{x}_1, \ldots, \mathfrak{x}_n), \quad F(\mathfrak{x}_1, \ldots, x_m) = \mathfrak{x} \qquad (P, F \in \Omega),$$

where $\mathfrak{x}, \mathfrak{x}_1, \mathfrak{x}_2, \ldots \in \{x_1, x_2, \ldots, x_k\}$, are called atomic formulas of signature Ω in $x_1, x_2, \ldots, x_k$.

The more complex expressions

$$P\big(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k)\big),$$
$$f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k),$$

where $f, g, f_1, \ldots, f_n$ are certain terms of signature Ω, $P \in \Omega$, are called *quasiatomic* formulas of signature Ω in the variables $x_1, \ldots, x_k$. Sometimes we shall for brevity write $P(f, g)$ for the formula $f = g$, with the understanding that P is the equality sign and is to be included in the signature of the formula.

Let $\mathscr{A}_1(x_1, \ldots, x_k), \ldots, \mathscr{A}_{s+1}(x_1, \ldots, x_k)$ be quasiatomic formulas of signature Ω (including equality) in the variables $x_1, \ldots, x_k$. Then formulas of the form

$$(\forall x_1 \ldots x_k)\, \mathscr{A}_1(x_1, \ldots, x_k) \tag{1}$$

are *identities* or *identical relations*, and formulas of the form

$$(\forall x_1 \ldots x_k)\, (\mathscr{A}_1 \& \ldots \& \mathscr{A}_s \to \mathscr{A}_{s+1}) \tag{2}$$

are *quasi-identities*.

In particular, if classes of algebras are being considered, then the signature Ω contains no predicate symbols and so identities and quasi-identities have the respective forms

$$(\forall x_1 \ldots x_k)\, \big(f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k)\big), \tag{3}$$

$$(\forall x_1 \ldots x_k)\, (f_1 = g_1 \& \ldots \& f_s = g_s \to f_{s+1} = g_{s+1}), \tag{4}$$

where $f, g, f_1, g_1, \ldots, f_{s+1}, g_{s+1}$ are terms of signature Ω in the variables $x_1, \ldots, x_k$.

From these definitions, it follows that identities and quasi-identities are special forms of Horn $\forall$-formulas (Section 7.5). We also point out that each identity (1) is equivalent to the quasi-identity

$$(\forall x_1 \ldots x_k)\, \big(x_1 = x_1 \to \mathscr{A}_1(x_1, \ldots, x_k)\big).$$

The converse does not hold. For example, the quasi-identity $x \leqq y \,\&\, y \leqq z \to x \leqq z$ is not equivalent to any identity since every identity is stable under homomorphism (Section 7.4) and the above quasi-identity is not.

For example, let $\mathfrak{A}_1 = \langle \{a, b, c\}, \leqq_1 \rangle$ be a model of type $\langle 2 \rangle$, in which the predicate $\leqq_1$ is defined as follows: $a \leqq_1 a$, $a \leqq_1 b$, $b \leqq_1 b$, $c \leqq_1 c$, $\neg a \leqq_1 c$, $\neg c \leqq_1 b$, $\neg b \leqq_1 a$, $\neg b \leqq_1 c$, $\neg c \leqq_1 a$.

Consider also the model $\mathfrak{A}_2 = \langle \{a, b, c\}, \leqq_2 \rangle$, where $a \leqq_2 a$, $a \leqq_2 b$, $b \leqq_2 b$, $c \leqq_2 c$, $b \leqq_2 c$, $\neg a \leqq_2 c$, $\neg c \leqq_2 b$, $\neg b \leqq_2 a$, and $\neg c \leqq_2 a$ (Fig. 1). The identity mapping of $\{a, b, c\}$ onto itself is a homomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. It is easy to check, however, that the quasi-identity $x \leqq y \,\&\, y \leqq z \to x \leqq z$ is true in $\mathfrak{A}_1$ but is false in $\mathfrak{A}_2$.

A class $\mathfrak{K}$ of systems is called a *variety* (*quasivariety*) if there is a collection $\mathfrak{S}$ of identities (quasi-identities) of signature Ω such that $\mathfrak{K}$ consists of those and only those systems of signature Ω in which all the formulas of $\mathfrak{S}$ are true. The collection $\mathfrak{S}$ is called the *defining set* of the variety or quasivariety. A variety (quasivariety) is said to be *finitely definable* if it has a finite defining set.

Since every identity is equivalent to a corresponding quasi-identity, every variety is a quasivariety. On the other hand, it is clear that a quasivariety is a particular type of universally axiomatizable class and so when studying the properties of quasivarieties, we may use the properties of universally axiomatizable classes developed in Section 7.2.

Any variety (quasivariety) is an axiomatizable class of algebras. Let us show that *if the variety (quasivariety) is finitely axiomatizable, then it is finitely definable.*

Indeed, suppose that the variety $\mathfrak{K}$ were definable by an infinite set of identities $\mathfrak{S}$ and hence by a certain formula Φ. This means that the

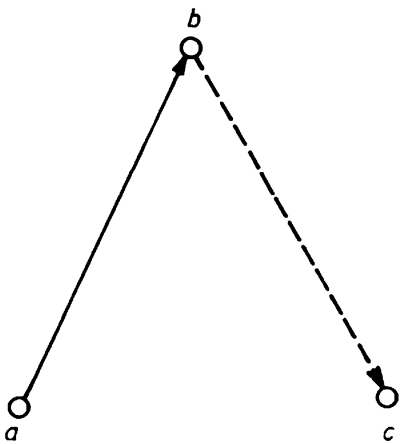

Fig. 1

formula Φ would be a consequence of the infinite system of formulas $\mathfrak{S}$. By the compactness theorem, the formula Φ would be a consequence of a suitable finite subcollection $\mathfrak{S}_f$ of $\mathfrak{S}$, and so the variety $\mathfrak{K}$ would be defined by the finite collection of identities $\mathfrak{S}_f$.

In particular, if a variety is definable by a finite number of quasi-identities, then it is definable by a finite number of identities.

To each signature Ω, there is up to isomorphism a unique trivial system $\mathfrak{A}_e$ consisting of just a single element e, for which

$$F(e, \ldots, e) = e, \quad P(e, \ldots, e) = \mathsf{T} \qquad (F, P, \in \Omega).$$

Hence, in a trivial system of signature Ω, all identities and quasi-identities of signature Ω are evidently true. In other words, *the trivial system of signature Ω belongs to every variety and every quasivariety of that signature.*

Consider now an arbitrary sequence $\mathfrak{K}_i$ $(i \in I)$ of varieties (or quasivarieties) of fixed signature Ω. Denote by $\mathfrak{S}_i$ the collection of all identities (quasi-identities) which define $\mathfrak{K}_i$, and let $\mathfrak{S}$ be the union of all $\mathfrak{S}_i$. It is clear that $\mathfrak{S}$ defines the intersection $\mathfrak{K}$ of the classes $\mathfrak{K}_i$. This intersection is nonempty since it contains the trivial system. Moreover, the class $\mathfrak{K}$ is a variety (or, correspondingly, a quasivariety).

We also note that among all quasivarieties of a given signature, there exists a largest. It is the variety of all systems of signature Ω. It may be defined by the identity

$$(\forall x)\,(x = x).$$

The smallest quasivariety of given signature is also a variety. It is obviously characterized by the identities

$$(\forall xy)\,(x = y), \quad (\forall x_1 \ldots x_k) P(x_1, \ldots, x_k) \qquad (P \in \Omega)$$

and consists only of the trivial system. Both of these varieties are called *trivial*; the first is sometimes called the *unit* variety, and the second, the *null* variety.

From their definition, we see that quasi-identities are special types of Horn axioms (Section 7.5). Classes of algebraic systems which are axiomatizable by Horn axioms are closed under filtered products (Section 8.4). Therefore, *an arbitrary filtered product of systems in a quasivariety belongs to that quasivariety.* In particular, the direct product of systems in an arbitrary quasivariety belongs to that quasivariety.

This implies that if a quasivariety is not the null quasivariety, then it contains infinite systems.

From the characteristic properties of universally axiomatizable classes developed in Sections 7.2. and 8.3, it is easy to deduce characteristic properties for quasivarieties.

Theorem 1 [46]. *A class of systems $\mathfrak{K}$ is a quasivariety if and only if $\mathfrak{K}$*

i) is locally closed;
ii) is multiplicatively closed;
iii) contains a trivial system.

The necessity of these conditions has been established above. Conversely, suppose that the class $\mathfrak{K}$ satisfies these conditions. By Theorem 1 of Section 7.2, i) implies that $\mathfrak{K}$ is universally axiomatizable. Theorem 2 of Section 7.5 and condition ii) imply that the $\mathfrak{K}$ is axiomatizable by means of Horn $\forall$-formulas. In Section 8.3 it was pointed out that every

Horn $\forall$-formula is equivalent to a conjunction of quasi-identities and, possibly, several formulas of the type

$$(\forall x_1 \dots x_n)\left(\neg P_1(f_1, \dots, f_m) \vee \dots \vee \neg P_s(f_1, \dots, f_m)\right), \qquad (5)$$

where $f_1, \dots, f_m$ are certain terms in the variables $x_1, \dots, x_n$. But in a trivial system, formulas such as (5) are false. Therefore, the class $\mathfrak{K}$ is axiomatizable by means of quasi-identities.

If instead of the results of Section 7 we use those of Section 8.3, we can prove a companion theorem.

Theorem 2. *A class of systems $\mathfrak{K}$ is a quasivariety if and only if $\mathfrak{K}$*

i) is ultraclosed;
ii) is hereditary;
iii) is multiplicatively closed;
iv) contains a trivial system.

The necessity of these conditions follows directly from Theorem 1 and Corollary 5 of Section 8.3. Therefore, suppose the class $\mathfrak{K}$ satisfies conditions i)—iv). According to Corollary 5 of Section 8.3, i) and ii) imply that $\mathfrak{K}$ is universally axiomatizable and is therefore locally closed. Applying Theorem 1, we conclude that $\mathfrak{K}$ is a quasivariety.

It was remarked above that quasivarieties are closed under filtered products. On the other hand. ultraproducts and direct products are but special cases of filtered products. Therefore, conditions i) and iii) in Theorem 2 can be replaced by the single condition of closure under filtered products.

Corollary 3. *A class of systems $\mathfrak{K}$ is a quasivariety if and only if $\mathfrak{K}$*

i) is closed under filtered products;
ii) is hereditary;
iii) contains a trivial system.

It is easy to show that the condition i) of closure under filtered products can not be replaced simply by multiplicative closure. For example, let p be a prime, and suppose $\mathfrak{G}$ is a group. Let $\mathfrak{G}^p$ denote the subgroup generated in $\mathfrak{G}$ by the $p^{-\text{th}}$ powers of all elements in $\mathfrak{G}$. Define $\mathfrak{G}_0 = \mathfrak{G}$ and $\mathfrak{G}_{n+1} = \mathfrak{G}_n^p$. A group $\mathfrak{G}$ is called a Δ_p-group if $\mathfrak{G}_\omega = \bigcap \mathfrak{G}_n$ is the trivial group. It is easy to check that the class of all Δ_p-groups satisfies conditions ii), iii), and iv) of Theorem 2. However, this class is not locally closed and hence, is not a quasivariety. Indeed, every finite cyclic group of order p^m is a Δ_p-group. Therefore, the locally cyclic group Z_{p^∞} constructed in Section 3.3 is locally embeddable in the class of Δ_p-groups. But this group itself is not a Δ_p-group since $Z_{p^\infty}^p = Z_{p^\infty}$.

Since the class of Δ_p-groups is hereditary, the group Z_{p^∞} is not embeddable in it and consequently, this class is not locally closed.

Corollary 3 allows us to obtain an explicit formula for the minimal quasivariety containing an arbitrarily given class $\mathfrak{K}$. Indeed, suppose that the given class $\mathfrak{K}$ has signature Ω. Denote by $Q\mathfrak{K}$ the collection of all quasi-identities of signature Ω which are true in every system in $\mathfrak{K}$. Then $KQ\mathfrak{K}$ will be the class of all systems of signature Ω which satisfy the quasi-identities $Q\mathfrak{K}$, i.e. it will be a quasivariety. It is clear that $\mathfrak{K} \subseteq KQ\mathfrak{K}$. It is also evident that every quasivariety which contains the class $\mathfrak{K}$ also contains the quasivariety $KQ\mathfrak{K}$. Thus, the quasivariety $KQ\mathfrak{K}$ is the least quasivariety containing $\mathfrak{K}$. It is sometimes called the *quasiprimitive closure* of $\mathfrak{K}$.

Adding the trivial system to $\mathfrak{K}$, we obtain a class which we shall denote by $\mathfrak{K}_e$.

The class of all systems isomorphic to filtered products of systems in a given class $\mathfrak{K}$ will be denoted by $F\mathfrak{K}$. By $SF\mathfrak{K}_e$ will be denoted the class of all subsystems of filtered products of $\mathfrak{K}_e$-systems (in general for any class $\mathfrak{L}$, $S\mathfrak{L}$ will denote the class of all subsystems of $\mathfrak{L}$-systems).

Theorem 4. *For every class of systems $\mathfrak{K}$, we have*

$$KQ\mathfrak{K} = SF\mathfrak{K}_e.$$

Since the class $KQ\mathfrak{K}$ is a quasivariety which contains the class $\mathfrak{K}$, properties i), ii), iii) of Corollary 3 imply that $SK\mathfrak{K}_e \subseteq KQ\mathfrak{K}$. On the other hand, $KQ\mathfrak{K}$ is the smallest quasivariety which contains $\mathfrak{K}$. Therefore, we need only prove that the class $SF\mathfrak{K}_e$ is a quasivariety, i.e. that this class has properties i), ii), and iii) of Corollary 3. Properties ii) and iii) are obvious. Let us prove i). Suppose we are given a sequence $\mathfrak{A}_\mu (\mu \in M)$ of systems of the class $SF\mathfrak{K}_e$. This means that for suitable $\mathfrak{K}_e$-systems $\mathfrak{C}_i$ and suitable filters $\langle D_\mu, I_\mu \rangle$*)

$$\mathfrak{A}_\mu \subseteq \prod \mathfrak{C}_i/D_\mu \qquad (i \in I_\mu).$$

Take an arbitrary filter D over M. On the basis of Section 8 (cf. Problem 1)

$$\prod \mathfrak{A}/_\mu D \subseteq \prod \mathfrak{C}_i/D^*, \tag{6}$$

where D^* is a suitable filter. But the relation (6) means that the class $SF\mathfrak{K}_e$ is closed with respect to filtered products.

In particular, Theorem 4 implies that *the quasiprimitive closure of a system $\mathfrak{A}$ consists of the trivial system and the subsystems of arbitrary filtered powers of $\mathfrak{A}$.*

*) The notation $\mathfrak{A} \subseteq \mathfrak{B}$ means that the system $\mathfrak{A}$ is isomorphic to a subsystem of $\mathfrak{B}$.

In discussing the problem of embedding systems of one ultraclosed class into systems of another such class (see Section 8.3), we saw that the embeddability conditions could be expressed as $\forall$-formulas. We now wish to show that the embeddability criteria can be expressed by a collection of quasi-identities if we consider not arbitrary ultraclosed classes, but quasivarieties.

Corollary 5. *The subclass $\mathfrak{K}$ of systems of a quasivariety $\mathfrak{K}_1$ of signature Ω_1 which are embeddable in a suitable system of a fixed quasivariety $\mathfrak{K}_2$ of signature $\Omega_2 \supseteq \Omega_1$ is itself a quasivariety.*

This means that the criteria for when a system of one quasivariety can be embedded in systems of another quasivariety can be expressed in terms of quasi-identities.

The proof is immediate since the subclass $\mathfrak{K}$ satisfies conditions i), ii) and iii) of Corollary 3.

As a typical example, consider the problem of embedding a semigroup into a group. The classes of semigroups and groups are quasivarieties and hence we can apply Corollary 5. Therefore, the conditions for embedding an arbitrary semigroup in some suitable group can be expressed as a collection of quasi-identities. For semigroups, they are of the form (4), where $f_1, g_1, \ldots, f_{m+1}, g_{m+1}$ are certain words in the alphabet $x_1, \ldots, x_n$. The simplest of these quasi-identities are, obviously, the formulas

$$x_1 x_2 = x_1 x_3 \to x_2 = x_3, \tag{7}$$

$$x_1 x_3 = x_2 x_3 \to x_1 = x_2, \tag{8}$$

which represent the laws for left and right cancellation, respectively.

A semigroup satisfying the quasi-identities (7) and (8) is called a *cancellation semigroup*. This implies in particular that semigroups with cancellation form a finitely axiomatizable quasivariety.

If the semigroup $\mathfrak{A}$ is commutative, then, as we know, the satisfiability of the quasi-identity (7) in $\mathfrak{A}$ is already sufficient for the embeddability of $\mathfrak{A}$ in a group. If $\mathfrak{A}$ is any noncommutative semigroup, then the satisfiability of the quasi-identities (7) and (8) in $\mathfrak{A}$ does not always guarantee the embeddability of $\mathfrak{A}$ in some group. In this connection, rather complicated quasi-identities were found which are necessary to the embeddability of an arbitrary semigroup in a group. The first of these (Mal'cev [39]) is the quasi-identity

$$ax = by \,\&\, cx = dy \,\&\, au = bv \to cu = dv.$$

Later on, complete systems of quasi-identities were found which were necessary and sufficient for the embeddability of an arbitrary

semigroup in a group (see Mal'cev [40], as well as Lambek [32] and Cohn [13], p. 265). These systems were infinite and it was proved (Mal'cev [41]) that they were not equivalent to any finite system of quasi-identities. In other words, a quasivariety of semigroups embeddable in a group is not finitely axiomatizable. This is all the more interesting since the original classes, the class of semigroups and the class of all groups, are finitely axiomatizable.

11.2 Defining Relations. To give an algebraic system in explicit form means to present its diagram. The diagram of an infinite system consists of an infinite number of formulas. Hence, the only possibility for presenting an infinite system is to indicate a way of constructing the formulas of its diagram. One such procedure of great theoretical importance is that of presenting an algebraic system by means of so-called defining relations. It amounts to the following.

Consider an algebraic system $\mathfrak{A} = \langle A, \Omega \rangle$ belonging to a fixed class of systems $\mathfrak{K}$. Select in $\mathfrak{A}$ an arbitrary generating set of elements a_i $(i \in I)$ (not necessarily distinct) and with each of these elements a_i, associate a special object symbol z_i. Next, let $\mathfrak{S}$ be a collection of quasi-atomic formulas of the form

$$f = g, \qquad P(f, g, \ldots, h), \tag{1}$$

where $f, g, \ldots, h$ are terms of signature Ω in the object variables $z_i (i \in I)$ and P is a predicate symbol of the signature.

The collection of generating symbols $z_i (i \in I)$ and the collection of formulas $\mathfrak{S}$ are said to be *defining collections* for $\mathfrak{A}$ in $\mathfrak{K}$ in the generators $a_i (i \in I)$, if:

1) all the formulas of $\mathfrak{S}$ are true in $\mathfrak{A}$ under the mapping $z_i \to a_i$ $(i \in I)$;

2) any formula of the form (1) which is true in $\mathfrak{A}$ under the mapping $z_i \to a_i$ $(i \in I)$ is a *consequence* in the class $\mathfrak{K}$ of the collection of formulas $\mathfrak{S}$.

A formula $\mathscr{A}(z_1, \ldots, z_n)$ is said to be consequence in the class $\mathfrak{K}$ of the collection of formulas $\mathfrak{S}$ if for any mapping $z_i \to m_i$ of the collection of symbols z_i into an arbitrary $\mathfrak{K}$-system $\mathfrak{M}$ under which every formula of $\mathfrak{S}$ is true in $\mathfrak{M}$, the formula $\mathscr{A}(m_1, \ldots, m_n)$ is also true in $\mathfrak{M}$.

The following theorem is a useful rephrasing of the definition just introduced.

Theorem 1. *Let there be given a class $\mathfrak{K}$ of algebraic systems, a set $\{z_i\}$ of symbols z_i $(i \in I)$ and an arbitrary collection $\mathfrak{S}$ of formulas of the form* (1). *An arbitrary $\mathfrak{K}$-system $\mathfrak{A}$ is isomorphic to a $\mathfrak{K}$-system for which the*

collection of generating symbols z_i and the collection of formulas $\mathfrak{S}$ are defining collections if and only if there exists a mapping $z_i \to a_i$ $(a_i \in \mathfrak{A},\ i \in I)$, with the following properties:

i) *the elements a_i $(i \in I)$ generate $\mathfrak{A}$;*

ii) *all the formulas in $\mathfrak{S}$ are true in $\mathfrak{A}$ under the mapping $z_i \to a_i$;*

iii) *if to a $\mathfrak{K}$-system $\mathfrak{M}$, there is a mapping $z_i \to m_i$ $(m_i \in \mathfrak{M},\ i \in I)$ for which all the formulas in $\mathfrak{S}$ are true in $\mathfrak{M}$, then there is a homomorphism φ of $\mathfrak{A}$ into $\mathfrak{M}$ taking a_i into m_i $(i \in I)$.*

Necessity. Suppose that the system $\mathfrak{A}$ is defined within $\mathfrak{K}$ by the collections z_i $(i \in I)$ and $\mathfrak{S}$. According to the definition, this means that there exists a mapping $\alpha: z_i \to a_i (a_i \in \mathfrak{A})$ which satisfies conditions 1), 2) and such that the elements a_i generate $\mathfrak{A}$. This means, in particular, that the mapping $z_i \to a_i$ has properties i) and ii). Let us show that α also has property iii). Suppose we are given a mapping $\beta: z_i \to m_i$ into some $\mathfrak{K}$-system $\mathfrak{M}$ for which all the formulas of $\mathfrak{S}$ are true in $\mathfrak{M}$. Consider the relation φ in $\langle \mathfrak{A}, \mathfrak{M} \rangle$ consisting of all possible pairs

$$\langle f(a_1, \ldots, a_n),\ f(m_1, \ldots, m_n) \rangle, \tag{2}$$

where f is an arbitrary term. We wish to show that φ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{M}$. Since the elements a_i generate $\mathfrak{A}$, any element of $\mathfrak{A}$ can be represented in the form $f(a_1, \ldots, a_n)$ and hence, the domain of the first component of the relation φ is $\mathfrak{A}$. Next, if for some f and g

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n), \tag{3}$$

then by 2), the formula

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_n) \tag{4}$$

is a *consequence* of the collection of formulas $\mathfrak{S}$ in the class $\mathfrak{K}$. But all formulas of $\mathfrak{S}$ are true in the system $\mathfrak{M}$ under the mapping $z_i \to m_i$ and $\mathfrak{M} \in \mathfrak{K}$. Hence, formula (4) is also true in $\mathfrak{M}$, i.e. in $\mathfrak{M}$ we have

$$f(m_1, \ldots, m_n) = g(m_1, \ldots, m_n). \tag{5}$$

This means that φ is a singlevalued mapping of $\mathfrak{A}$ into $\mathfrak{M}$. In a similar way, we show that if in $\mathfrak{A}$ some relation of the form

$$P\big(f(a_1, \ldots, a_n), \ldots, h(a_1, \ldots, a_n)\big) \tag{6}$$

is true, where P is a predicate symbol of the signature, then the corresponding relation

$$P\big(f(m_1, \ldots, m_n), \ldots, h(m_1, \ldots, m_n)\big) \tag{7}$$

is true in $\mathfrak{M}$.

From (5) and (7), it follows that the mapping φ is a homomorphism. Since pairs of the form (2) belong to φ, we have $\langle a_i, m_i \rangle \in \varphi$, that is, $a_i\varphi = m_i (i \in I)$, as required.

Sufficiency. Suppose that the system $\mathfrak{A}$ satisfies conditions i), ii), iii) and suppose that for certain terms $f, g, \ldots, h$ relation (3) or relation (6) is true in $\mathfrak{A}$. We must show that formula (4) (or correspondingly, formula (7)) is then a consequence in the class $\mathfrak{K}$ of the collection of formulas $\mathfrak{S}$. In other words, we must show that for every $\mathfrak{K}$-system $\mathfrak{M}$, for any mapping $z_i \to m_i$ $(m_i \in \mathfrak{M},\ i \in I)$ under which all the formulas of $\mathfrak{S}$ are true in $\mathfrak{M}$, the formula (4) (or correspondingly, formula (7) is also true in $\mathfrak{M}$. According to iii), there exists under the above assumptions a homomorphism φ of the system $\mathfrak{A}$ into the system $\mathfrak{M}$ for which $a_i\varphi = m_i$ $(i \in I)$. But under homomorphism, true quasi-atomic formulas remain true and hence, if (3) or (6) is true, then (4) or, respectively, (7) is true.

We point out two corollaries which follow directly from Theorem 1.

Corollary 2. *If in a class $\mathfrak{K}$ there exist systems $\mathfrak{A}$ and $\mathfrak{B}$ which have a fixed collection of generating symbols $z_i (i \in I)$ and the same collection of quasi-atomic formulas $\mathfrak{S}$ for their defining collections, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

Indeed, according to Theorem 1 there are mappings

$$z_i \to a_i \quad (a_i \in \mathfrak{A},\ i \in I), \qquad z_i \to b_i \quad (b_i \in \mathfrak{B}, i \in I) \tag{8}$$

such that the elements a_i generate $\mathfrak{A}$, the elements b_i generate $\mathfrak{B}$ and for suitable homomorphisms

$$\varphi\colon \mathfrak{A} \to \mathfrak{B}, \qquad \psi\colon \mathfrak{B} \to \mathfrak{A}$$

we have $a_i\varphi = b_i$ and $b_i\psi = a_i$. Hence

$$a_i\varphi\psi = a_i, \qquad b_i\psi\varphi = b_i \qquad (i \in I). \tag{9}$$

Since $\varphi\psi$ and $\psi\varphi$ are homomorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ into themselves, (9) implies

$$f(a_1, \ldots, a_n)\varphi\psi = f(a_1, \ldots, a_n),$$
$$f(b_1, \ldots, b_n)\psi\varphi = f(b_1, \ldots, b_n),$$

i.e. $\varphi\psi$ and $\psi\varphi$ are identity mappings, $\psi = \varphi^{-1}$ and φ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Corollary 3. *Suppose that the collection of generating symbols z_i $(\alpha \in I)$ and quasi-atomic formulas $\mathfrak{S}$ define a system $\mathfrak{A}$ within the class $\mathfrak{K}$ and a system $\mathfrak{B}$ within a subclass $\mathfrak{L} \subseteq \mathfrak{K}$. Then the system $\mathfrak{B}$ is a homomorphic image of the system $\mathfrak{A}$.*

By hypothesis, there exist mappings (8) under which all the formulas of $\mathfrak{S}$ are true in $\mathfrak{A}$ and $\mathfrak{B}$ and the collections of elements $\{a_i\}, \{b_i\}$ generate these systems. From the fact $\mathfrak{B} \in \mathfrak{L}$, it follows that $\mathfrak{B} \in \mathfrak{K}$, and hence condition iii) of Theorem 1 implies the existence of a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ such that $a_i\varphi = b_i$ $(i \in I)$. Since the image $\mathfrak{A}\varphi$ of the system $\mathfrak{A}$ in $\mathfrak{B}$ contains the elements b_i which generate $\mathfrak{B}$, φ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

If the class $\mathfrak{K}$ is arbitrary, it may happen that a certain collection of generating symbols and quasi-atomic formulas defines no system in $\mathfrak{K}$. For example, suppose $\mathfrak{K}$ consists only of systems isomorphic to the semigroup

$$\mathfrak{G} = \langle \{2, 3, 4, \ldots\}, \cdot \rangle.$$

It is clear that this semigroup has no elements z_1 and z_2 which satisfy the condition $z_1 \cdot z_2 = z_1$. Therefore, the generating symbols z_1 and z_2 and formula $z_1 \cdot z_2 = z_1$ define no system in $\mathfrak{K}$.

In which classes will each collection of quasi-atomic formulas serve as a defining collection for some system? The answer to this question is furnished by the following basic theorem.

Theorem 4. *If a class $\mathfrak{K}$ of algebraic systems is hereditary, multiplicatively closed, and contains a trivial system (in particular, if $\mathfrak{K}$ is a quasivariety), then any collection of generating symbols $z_i (i \in I)$ and any collection $\mathfrak{S}$ of quasi-atomic formulas in the variables z_i having the signature of the class $\mathfrak{K}$ defines some $\mathfrak{K}$-system.*

Suppose $\mathfrak{m} = \max(|\Omega|, |I|, \aleph_0)$, where $|\Omega|$, $|I|$ and $\aleph_0$ are the respective cardinalities of the signature of the class $\mathfrak{K}$, the collection of generating symbols z_i, and the set of natural numbers. By Section 6 (see Problems and Complements, Item 2), every algebraic system with a generating set of cardinality $\leqq |I|$ is itself of cardinality not greater than $\mathfrak{m}$.

Denote by M the set of all possible pairwise nonisomorphic $\mathfrak{K}$-systems each of whose cardinality does not exceed $\mathfrak{m}$. It is clear that M is a set of well-defined cardinality not larger than a certain bound depending only on $\mathfrak{m}$ and $|\Omega|$.

Consider the family $\alpha_\lambda (\lambda \in J)$ of all possible mappings

$$\alpha_\lambda\colon z_i \to a_i^\lambda \qquad (a_i^\lambda \in \mathfrak{A}_\lambda,\ \mathfrak{A}_\lambda \in M) \tag{10}$$

of the collection of generating symbols z_i into the systems of M such that under α_λ all the formulas of $\mathfrak{S}$ are true in $\mathfrak{A}_\lambda$. Since for the same system $\mathfrak{A}$ there may be several distinct mappings $z_i \to a_i$ under which the formulas of $\mathfrak{S}$ are true, it is possible that $\mathfrak{A}_\lambda = \mathfrak{A}_\mu$ in (10) for certain $\lambda \neq \mu$ in J.

The family of mappings α_λ is clearly nonempty since it contains the trivial system $\mathfrak{A}_e = \langle \{e\}, \Omega \rangle$ in which all the formulas of $\mathfrak{S}$ as well as all quasi-atomic formulas are always true under the mapping $z_i \to e$.

Consider the direct product

$$\mathfrak{B} = \prod \mathfrak{A}_\lambda \qquad (\lambda \in J).$$

Since all its factors belong to the class $\mathfrak{K}$, which is multiplicatively closed, we have $\mathfrak{B} \in \mathfrak{K}$. The mappings (10) induce the mapping

$$\alpha\colon z_i \to \mathfrak{B} \qquad (i \in I).$$

By Section 7.5, the quasi-atomic formulas are multiplicatively stable. The formulas of $\mathfrak{S}$ are quasi-atomic and are true in the factors $\mathfrak{A}_\lambda$ under the mappings α_λ. Therefore, the formulas of $\mathfrak{S}$ are true in $\mathfrak{B}$ under the mapping α.

Set $z_i\alpha = a_i$ and let $\mathfrak{A}$ denote the subsystem of $\mathfrak{B}$ generated in $\mathfrak{B}$ by the elements $a_i (i \in I)$. Since the formulas in $\mathfrak{S}$ are quantifier-free and since they are true in the system $\mathfrak{B}$ under the mapping $\alpha\colon z_i \to a_i$, it follows that they also hold in the subsystem $\mathfrak{A}$.

We wish to show that the generating symbols z_i and the formulas of the collection $\mathfrak{S}$ define precisely the system $\mathfrak{A}$ in the class $\mathfrak{K}$. Indeed, this system is a subsystem of a $\mathfrak{K}$-system $\mathfrak{B}$ and hence belongs itself to $\mathfrak{K}$. The elements a_i generate $\mathfrak{A}$ and under the mapping $\alpha\colon z_i \to a_i$ the formulas of $\mathfrak{S}$ are true in $\mathfrak{A}$. It remains to show that the mapping α has property iii) of Theorem 1.

Suppose we are given a $\mathfrak{K}$-system $\mathfrak{M}$ and a mapping $z_i \to m_i$ ($m_i \in \mathfrak{M}$, $i \in I$) under which the formulas of $\mathfrak{S}$ are true in $\mathfrak{M}$. Let $\mathfrak{N}$ denote the subsystem generated in $\mathfrak{M}$ by the elements m_i. The system $\mathfrak{N}$ belongs to $\mathfrak{K}$ and under the mapping $z_i \to m_i$ the formulas of $\mathfrak{S}$ are true in $\mathfrak{N}$. We must find a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{N}$ such that $a_i\varphi = m_i$ $(i \in I)$.

The set of elements m_i generates $\mathfrak{N}$ and its cardinality is not greater than $|I|$. Therefore, up to isomorphism, $\mathfrak{N}$ coincides with some system $\mathfrak{A}_j$ and the mapping $z_i \to m_i$ coincides with the mapping $\alpha_j\colon z_i \to a_i^j$. Thus, it is sufficient for us to find a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{A}_j$ satisfying $a_i\varphi = a_i^j$ $(i \in I)$.

The mapping $\alpha\colon z_i \to \mathfrak{B}$ is, by hypothesis, induced by the mappings (10), and hence for the projection π_j, we have

$$\pi_j\colon \mathfrak{A} \to \mathfrak{A}_j, \qquad a_i\pi_j = a_i^j,$$

i.e. π_j is just the required homomorphism of $\mathfrak{A}$ into $\mathfrak{A}_j$.

In Section 11.3 (Theorem 5), it will be shown that Theorem 4 has a converse. At this point, we shall prove a series of results which will permit a rather complete description of the structure of systems presented by defining formulas.

Theorem 5. (Dyck's Theorem). *Suppose that in a class $\mathfrak{K}$, two collections of defining relations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ with the same generating symbols $\{z_i | i \in I\}$ define the systems $\mathfrak{A}$ and $\mathfrak{B}$, respectively. If all the formulas of $\mathfrak{S}_1$ are consequences in $\mathfrak{K}$ of formulas of $\mathfrak{S}_2$, in particular, if $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, then $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. If the collections $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equivalent in $\mathfrak{K}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

According to Theorem 1, there exist mappings $z_i \to a_i (a_i \in \mathfrak{A})$ and $z_i \to b_i (b_i \in \mathfrak{B})$ such that the collections $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$, respectively, generate the systems $\mathfrak{A}$ and $\mathfrak{B}$; also, with the given values for the z_i, the formulas of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are true in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Since the formulas belonging to $\mathfrak{S}_1$ are consequences of those in $\mathfrak{S}_2$, all the formulas of $\mathfrak{S}_1$ are also true in $\mathfrak{B}$. By iii) of Theorem 1, it now follows that there is a homomorphism φ of $\mathfrak{A}$ into $\mathfrak{B}$ taking a_i into $b_i (i \in I)$. Since the elements b_i generate $\mathfrak{B}$, φ is the desired homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are equivalent, then along with the homomorphism φ: $\mathfrak{A} \to \mathfrak{B}$, there also exists a homomorphism ψ: $\mathfrak{B} \to \mathfrak{A}$ taking b_i to a_i. Since the collections $\{a_i \mid i \in I\}$ and $\{b_i \mid i \in I\}$ generate $\mathfrak{A}$ and $\mathfrak{B}$ and $a_i \varphi \psi = a_i$, $b_i \psi \varphi = b_i$, we have $\psi = \varphi^{-1}$ and $\mathfrak{A} \cong \mathfrak{B}$.

Dyck's theorem indicates how to vary the collection $\mathfrak{S}$ of defining relations while keeping the generating set $\{z_i \mid i \in I\}$ fixed so that the system $\mathfrak{A}$ being defined remains the same. We now consider how $\mathfrak{S}$ must be changed when the generators z_i are changed so that the system defined remains the same.

Theorem 6 (Tietze's Theorem). *In a class $\mathfrak{K}$, let the collection of generators $\{z_i \mid i \in I\}$ and defining relations $\mathfrak{S}$ define system $\mathfrak{A}$. If we add new symbols x_α to the collection of generators and adjoin to the collection $\mathfrak{S}$ any relations of the form $x_\alpha = f_\alpha(z_i, \ldots, z_k)$, we obtain a collection of generators and relations which define the same system $\mathfrak{A}$. Conversely, suppose that the collection $\mathfrak{S}$ contains relations of the form $z_\alpha = f_\alpha(z_i, \ldots, z_k)$ $(\alpha \in J)$, where the symbols z_α do not occur in the terms f_α. In each of the remaining relations in $\mathfrak{S}$ substitute for each embedment of z_α the term $f_\alpha(z_i, \ldots, z_k)$. This yields a new collection of relations $\mathfrak{S}^*$ in which the symbols z_α no longer occur. On deleting the symbols z_α $(\alpha \in J)$ from the list of generators, we obtain a smaller collection of generators z_λ $(\lambda \in I - J)$, which together with the collection of relations $\mathfrak{S}^*$ again defines the system $\mathfrak{A}$ in the class $\mathfrak{K}$.*

Let us prove the first assertion. Consider any mapping $z_i \to a_i$ $(a_i \in \mathfrak{A})$ which has properties i)—iii) of Theorem 1. The additional symbols x_α are to be mapped into the respective elements $b_\alpha = f_\alpha(a_i, \ldots, a_k) \in \mathfrak{A}$. Suppose $z_i \to m_i$, $x_\alpha \to n_\alpha$ is a mapping of the symbols z_i, x_α into an arbitrary $\mathfrak{K}$-system $\mathfrak{M}$ under which the relations in $\mathfrak{S}$ and the relations $x_\alpha = f_\alpha(z_i, \ldots, z_k)$ are true in $\mathfrak{M}$. By iii), there is a homo-

morphism $\varphi: \mathfrak{A} \to \mathfrak{M}$ taking a_i into m_i $(i \in I)$. Since $f_\alpha(a_{i_1}, \ldots, a_{i_k})\varphi = f_\alpha(m_{i_1}, \ldots, m_{i_k})$, the homomorphism φ takes b_α into n_α, as required.

We now prove the second assertion. Again, consider a mapping $z_s \to a_s$ $(a_s \in \mathfrak{A}, s \in I)$, which satisfies conditions i)—iii). Since $a_\alpha = f_\alpha(a_{i_1}, \ldots, a_{i_k})$, the smaller collection of elements a_λ $(\lambda \in I - J)$ will still generate the system $\mathfrak{A}$. It is clear that under the mapping $z_\lambda \to a_\lambda$ $(\lambda \in I - J)$ all the formulas of the collection $\mathfrak{S}^*$ are true in $\mathfrak{A}$. Suppose $z_\lambda \to m_\lambda$ $(\lambda \in I - J)$ is a mapping of the smaller list of generating symbols into some $\mathfrak{K}$-system $\mathfrak{M}$ such that the formulas of $\mathfrak{S}^*$ are true in $\mathfrak{M}$. In $\mathfrak{M}$ take the elements $m_\alpha = f_\alpha(m_{i_1}, \ldots, m_{i_k})$ and consider the mapping $z_s \to m_s$ $(s \in I)$. Under this mapping all the formulas in $\mathfrak{S}$ are true in $\mathfrak{M}$ and hence, there exists a homomorphism $\varphi: \mathfrak{A} \to \mathfrak{M}$ taking a_s into m_s $(s \in I)$.

An algebraic system $\mathfrak{A}$ is said to be *finitely presented* or *finitely definable* in the class $\mathfrak{K}$ if a finite set of generators and a finite set of defining relations $\mathfrak{S}$ exist which determine $\mathfrak{A}$ in the class $\mathfrak{K}$.

A generating set of elements of a system $\mathfrak{A}$ is called a *basis* for $\mathfrak{A}$ if no proper subset of this collection generates $\mathfrak{A}$. Theorem 2 of Section 6.1 implies that for a finitely generated algebraic system, every set of generators includes some basis. From the definitions, every finitely presented system is finitely generated. Below, we shall show by examples that the converse is false. Therefore, one might think it possible for a finitely generated system $\mathfrak{A}$ to have a finite set of defining relations in one basis and none in another. However, the following corollary to Tietze's theorem refutes this.

Corollary 7. *If in a class $\mathfrak{K}$, a finitely generated system $\mathfrak{A}$ has a finite set of defining relations in terms of some basis, then in $\mathfrak{K}$, the system $\mathfrak{A}$ has a finite set of defining relations in terms of any basis for $\mathfrak{A}$.*

Let $z_1, \ldots, z_s$ and $x_1, \ldots, x_t$ be two bases for the system $\mathfrak{A}$. Let $\mathfrak{S}$ be a set of defining relations in the first basis. By hypothesis

$$x_j = f_j(z_1, \ldots, z_s) \quad (j = 1, \ldots, t), \tag{11}$$

$$z_i = g_i(x_1, \ldots, x_t) \quad (i = 1, \ldots, s), \tag{12}$$

where f_j and g_i are suitable terms in the indicated object symbols. According to the Tietze theorem, the system $\mathfrak{A}$ can be presented by the generators $z_1, \ldots, z_s, x_1, \ldots, x_t$ and defining relations $\mathfrak{S}_1$ consisting of $\mathfrak{S}$ and the formulas (11). The formulas (12) are consequences of $\mathfrak{S}_1$ in the class $\mathfrak{K}$. Hence, by Dyck's theorem, their adjunction to the collection $\mathfrak{S}_1$ results in a set of defining relations $\mathfrak{S}_2$ which still determines the same system $\mathfrak{A}$. To the collection $\mathfrak{S}_2$ we apply the second part of Tietze's theorem;

if in each formula of $\mathfrak{S}_2$ we substitute for the symbols z_i corresponding terms $g_i(x_1, \ldots, x_t)$, then we shall obtain a collection of relations $\mathfrak{S}_2^*$ which define $\mathfrak{A}$ in the basis $x_1, \ldots, x_t$. If the initial collection $\mathfrak{S}$ is finite, then $\mathfrak{S}_2^*$ will also be finite.

The theorems of Dyck and Tietze yield the following procedure for going from a system of defining relations $\mathfrak{S}$ in one set of generating symbols z_i $(i \in I)$ to any other system of defining formulas $\mathfrak{S}^*$ in any other generating set z_j $(j \in J)$:

A) To the symbols z_i, add new arbitrary symbols z_j $(j \in J, J \cap I = \varnothing)$, and to the collection $\mathfrak{S}$, add an arbitrary number of formulas of the form (11) (one formula for each new variable). Let $\mathfrak{S}_1$ be the new collection.

B) To the collection $\mathfrak{S}_1$, add an arbitrary number of formulas which are consequences of $\mathfrak{S}_1$ in the class $\mathfrak{K}$, and then keep any such part $\mathfrak{S}_2$ of the new collection from which the formulas to be deleted follow.

C) From the enlarged collection of generating symbols z_λ $(\lambda \in I \cup J)$, exclude the set of symbols $z_r (r \in R)$ to each of which there is a formula in $\mathfrak{S}_2$ of the form

$$z_r = f_r(z_i, \ldots, z_k) \qquad (i, \ldots, k \notin R). \tag{13}$$

For every formula in $\mathfrak{S}_2$, replace each occurrence of a z_r $(r \in R)$ by its expression in (13).

The collection $\mathfrak{S}^*$ of new formulas will define system $\mathfrak{A}$ in the generators z_i $(i \notin R)$. From the proof of Corollary 7, it follows that the transformations A)—C) can be used to go from one system of generators and defining relations to any other system of generators and defining relations.

The algorithmic meaning of transformations A) and C) is clear. What is the nature of the transformations B)? In order to answer this question, we shall suppose that the class $\mathfrak{K}$ is a quasivariety defined by the quasi-identities

$$(\forall x_1 \ldots x_{n_\alpha}) \left(P_{\alpha 1}(f(x)) \,\&\, \ldots \,\&\, P_{\alpha p_\alpha}(f(x)) \to P_{\alpha 0}(f(x))\right). \tag{14}$$

Here $P_{\alpha i}(f(x))$ $(\alpha \in U)$ denotes a formula of the form

$$P(f_1(x_1, \ldots, x_{n_\alpha}), \ldots, f_k(x_1, \ldots, x_{n_\alpha})),$$

where P is a signature predicate symbol or the equality sign and $f_1, \ldots, f_k$ are terms of signature Ω in the variables $x_1, \ldots, x_{n_\alpha}$. The formulas (14) may contain the symbol for equality taken in the absolute sense. We relativize it, i.e. we replace the usual $=$ sign in the above formulas by

the symbol $\equiv$ and we introduce the formulas

$$\left.\begin{aligned}
&x_1 \equiv x_1, \quad x_1 \equiv x_2 \to x_2 \equiv x_1,\\
&x_1 \equiv x_2 \,\&\, x_2 \equiv x_3 \to x_1 \equiv x_3,\\
&x_1 \equiv x_{n+1} \,\&\, \ldots \,\&\, x_n \equiv x_{n+n}\\
&\qquad\qquad \to F(x_1, \ldots, x_n) \equiv F(x_{n+1}, \ldots, x_{n+n}),\\
&x_1 \equiv x_{n+1} \,\&\, \ldots \,\&\, x_n \equiv x_{n+n} \,\&\, P(x_1, \ldots, x_n)\\
&\qquad\qquad \to P(x_{n+1}, \ldots, x_{n+n}).
\end{aligned}\right\} \tag{15}$$

Formulas (15) have the same form as (14). Therefore, we shall now assume that the system (14) is a complete system of formulas (including formulas (15)) with the relativized equality sign which defines the quasivariety $\mathfrak{K}$.

Now suppose we are given an arbitrary collection of generating symbols z_i $(i \in I)$ and an arbitrary collection $\mathfrak{S}$ of quasi-atomic formulas in the variables z_i. If in the quantifier-free part of (14) we replace the variables $x_1, \ldots, x_{n_\alpha}$ with arbitrary terms $h_1(z_1, \ldots, z_s), \ldots, h_{n_\alpha}(z_1, \ldots, z_s)$, we obtain a formula which we shall write in the brief form

$$P_{\alpha 1}\big(f(h(z))\big) \,\&\, \ldots \,\&\, P_{\alpha p_\alpha}\big(f(h(z))\big) \to P_{\alpha 0}\big(f(h(z))\big). \tag{16}$$

Let $\mathfrak{T}$ be the collection of all formulas (16) which can be obtained from the formulas (14) in the indicated manner.

Next we construct a sequence $\mathfrak{S}_0 \subseteq \mathfrak{S}_1 \subseteq \ldots \subseteq \mathfrak{S}_n \subseteq \ldots$ of collections of quasi-atomic formulas in the z_i. By definition, we set $\mathfrak{S}_0 = \mathfrak{S}$. Suppose that for a certain natural number $n \geqq 0$, the collection $\mathfrak{S}_n$ has been defined. In the collection $\mathfrak{T}$, we look for those formulas (16) in which all the subformulas $P_{\alpha t}\big(f(h(z))\big)$ $(t = 1, \ldots, p_\alpha)$ in the antecedents are contained in $\mathfrak{S}_n$. The right-hand sides $P_{\alpha 0}\big(f(h(z))\big)$ of all these formulas are formal consequences of the formulas in $\mathfrak{S}_n$ and the formulas (14). The collection obtained by adjoining them to $\mathfrak{S}_n$ is by definition the collection $\mathfrak{S}_{n+1}$.

Denote by $\mathfrak{S}_\omega$ the union of the collections $\mathfrak{S}_n$. Let us show that $\mathfrak{S}_\omega$ is the set of all quasi-atomic formulas in the variables z_i which are formal consequences in $\mathfrak{K}$ of the collection $\mathfrak{S}$.

Indeed, from the construction of the collection $\mathfrak{S}_n$ it is clear that all the formulas in $\mathfrak{S}_\omega$ are consequences of $\mathfrak{S}$ in $\mathfrak{K}$. Therefore, we need only show that the quasi-atomic formulas which are not in $\mathfrak{S}_\omega$ are not consequences of $\mathfrak{S}$ in the class $\mathfrak{K}$. In other words, we must construct a system $\mathfrak{A}$ defined in $\mathfrak{K}$ by the generating symbols z_i and the formulas of $\mathfrak{S}$

and show that all the quasi-atomic formulas which are true in $\mathfrak{A}$ under the canonical mapping $z_i \to a_i$ $(a_i \in \mathfrak{A},\ i \in I)$ belong to $\mathfrak{S}_\omega$.

Let B denote the collection of all terms of signature Ω in the variables z_i. We define in B free term operations (Section 6.1) and say that a relation $P(f_1, \ldots, f_k)$ $(P \in \Omega;\ f_1, \ldots, f_k \in B)$ is true if the formula $P(f_1, \ldots, f_k)$ is contained in $\mathfrak{S}_\omega$. This yields an algebraic system $\mathfrak{B} = \langle B, \Omega \rangle$. Elements $f, g \in B$ are called equivalent if the collection $\mathfrak{S}_\omega$ contains the formula $f \equiv g$. Since in the construction of $\mathfrak{S}_\omega$ the formulas (15) were taken into account, the relation $\equiv$ is a congruence on the system $\mathfrak{B}$ and we can form the quotient system $\mathfrak{B}/\equiv$.

The canonical epimorphism (onto-homomorphism) $\sigma: \mathfrak{B} \to \mathfrak{B}/\equiv$ yields a mapping $\sigma: z_i \to z_i\equiv$ $(i \in I)$ under which all the formulas of $\mathfrak{S}_\omega$ are true, and the quasi-atomic formulas in the z_i $(i \in I)$ which are not in $\mathfrak{S}_\omega$ are false in $\mathfrak{B}/\equiv$. It remains only to show that $\mathfrak{B}/\equiv \in \mathfrak{K}$, i.e. that all the quasi-identities (14) are true in $\mathfrak{B}/\equiv$ (with absolute equality sign).

Suppose, to the contrary, that one of the quasi-identities (14) is false in $\mathfrak{K}/\equiv$. Then $\mathfrak{B}/\equiv$ has elements $h_1\equiv, \ldots, h_{n_a}\equiv$ $(h_1, \ldots, h_{n_a} \in B)$, for which the formulas

$$P_{at}\big(f(h\equiv)\big) \qquad (t = 1, \ldots, n_a) \tag{17}$$

are true in $\mathfrak{B}/\equiv$ but the formula $P_{a0}(f(h\equiv))$ is false. But the satisfiability of formulas (17) in the system $\mathfrak{B}/\equiv$ is equivalent to the satisfiability of the formulas

$$P_{at}\big(f(h)\big) \qquad (t = 1, \ldots, n_a) \tag{18}$$

in the system $\mathfrak{B}$ which by definition means that the formulas (18) belong to the collection $\mathfrak{S}_\omega$. Hence, they belong to $\mathfrak{S}_n$ for some suitable number n. By formula (16), this implies that the formula $P_{a0}\big(f(h)\big)$ belongs to $\mathfrak{S}_{n+1}$ and is therefore true in $\mathfrak{B}/\equiv$, in contradiction to the above assumption.

Thus, $\mathfrak{S}_\omega$ is the collection of all quasi-atomic formulas in the variables z_i which follow from the collection $\mathfrak{S}$ in the class $\mathfrak{K}$. In particular, according to the definitions, this means that $\mathfrak{B}/\equiv$ is a system defined in the class $\mathfrak{K}$ by the generating symbols z_i and defining formulas $\mathfrak{S}$. The above process of constructing $\mathfrak{B}/\equiv$ is often referred to as the *standard construction of an algebraic system from its generators and defining relations.*

It should be pointed out that our solution to the problem of constructing the quasi-atomic formulas which follow from given quasi-identities and quasi-atomic formulas is a special case of Hilbert's algorithm for constructing all first-order formulas which are identically true. However, since we are interested only in formulas of a very special

form, the above construction is much simpler than Hilbert's general algorithm.

In conclusion, let us examine several examples of actual construction of an algebraic system from its defining formulas.

a) Let the class $\mathfrak{K}$ be the quasivariety of partially ordered sets defined by the quasi-identities

$$x \leqq x, \quad x \leqq y \,\&\, y \leqq x \to x = y, \quad x \leqq y \,\&\, y \leqq z \to x \leqq z. \tag{19}$$

The signature of this class does not contain any operator symbols and so the quasi-atomic formulas are only of the form $z_i = z_j$ or $z_i \leqq z_j$. Therefore, in order to present a partially ordered set by means of defining relations, one must specify the collection of generating symbols z_i $(i \in I)$ and a collection $\mathfrak{S}$ of formulas of the form

$$z_i = z_j, \quad z_k \leqq z_l \quad (i, j, k, l \in I). \tag{20}$$

To the collection (20), we next apply procedures C) and B) indicated above. Namely, if (20) includes any equations, we pick out those of the form

$$z_i = z_k \quad (i \in I_1,\ k \in I_2,\ I_1 \cap I_2 = \varnothing).$$

We delete all the symbols z_i $(i \in I_1)$ from the list of generators and in all formulas of (20) replace each z_i by the corresponding z_k.

We check to see if the new collection (20) has any pairs of the form $z_i \leqq z_j$, $z_j \leqq z_i$ or $z_i \leqq z_j$, $z_j \leqq z_k$. If so, then we add the formulas $z_i = z_j$ or $z_i \leqq z_k$ to the collection (20) and we again apply the deletion process.

To normalize these deletion processes, one could, for example, well order the index set I and delete the symbols with larger indices. The collection of all remaining symbols and the collection of all resulting formulas of the form $z_i \leqq z_j$ involving these symbols would define the desired model.

b) Suppose that the class $\mathfrak{L}$ has a signature consisting of the predicate symbol $\leqq$ and a one-place operator symbol $'$. Suppose that $\mathfrak{L}$ is defined by the axioms (19) and the additional quasi-identity

$$x' \leqq x. \tag{21}$$

Consider the system $\mathfrak{A}$ defined in $\mathfrak{L}$ by generating symbols a and b and defining formulas

$$a^{(i)} \leqq b^{(i)} \ (a^{(i+1)} = a^{(i)\prime};\ i = 1, 2, \ldots). \tag{22}$$

It is easy to see that the relations of the system on the left in Fig. 1 follow from the formulas in (22) and that the quasi-identities (19) and (21)

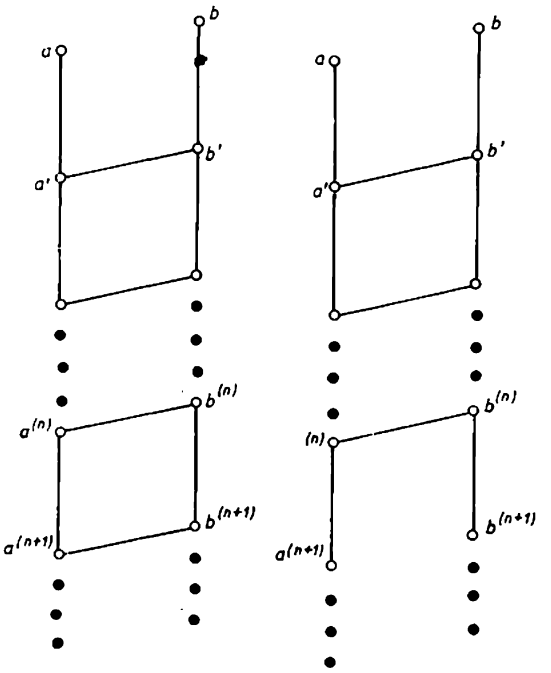

Fig. 1

are also true in this system. Therefore, that system is defined in $\mathfrak{L}$ by the formulas in (22).

In a similar way, one sees that the system on the right in Fig. 1 is defined in $\mathfrak{L}$ by the formulas

$$a^{(i)} \leqq b^{(i)} \qquad (i = 1, \ldots, n).$$

From this it is clear that the system $\mathfrak{A}$ defined by the infinite system of formulas (22) can not be defined in $\mathfrak{L}$ by any finite collection of quasi-atomic formulas.

11.3 Replicas. Suppose $\mathfrak{K}$ is an arbitrary class of algebraic systems of signature Ω and $\mathfrak{A}$ is some system of signature Ω not necessarily in $\mathfrak{K}$. A homomorphism α of $\mathfrak{A}$ *onto* a $\mathfrak{K}$-system $\mathfrak{A}_1$ will be called a $\mathfrak{K}$-*morphism* if for every homomorphism γ of $\mathfrak{A}$ into an arbitrary $\mathfrak{K}$-system $\mathfrak{C}$ there exists a homomorphism ξ of $\mathfrak{A}_1$ *onto* the system $\mathfrak{C}$ such that $\gamma = \alpha\xi$ i.e. the diagram in Fig. 1 is commutative. Every $\mathfrak{K}$-morphic image of the system $\mathfrak{A}$ is called a *replica* of $\mathfrak{A}$ in the class $\mathfrak{K}$ ($\mathfrak{K}$-replica) and is denoted by $\mathfrak{A}_{\mathfrak{K}}$.

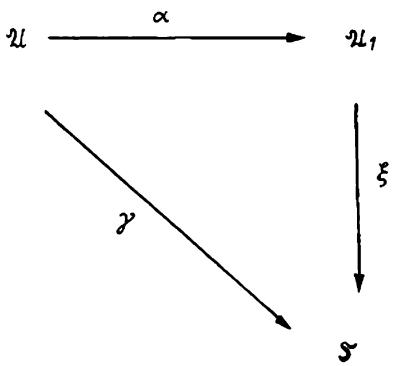

Fig. 1

From this definition, if follows that the identity mapping of every $\mathfrak{K}$-system onto itself is a $\mathfrak{K}$-morphism, and hence every $\mathfrak{K}$-system is its own replica in $\mathfrak{K}$.

We note several other direct consequences of the definition of replica.

Theorem 1. *Let $\mathfrak{K}$ be any class of systems and let $\alpha: \mathfrak{A} \to \mathfrak{A}_{\mathfrak{K}}$ and $\beta: \mathfrak{B} \to \mathfrak{B}_{\mathfrak{K}}$ be $\mathfrak{K}$-morphisms of systems onto their $\mathfrak{K}$-replicas. If there is an epimorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$, then there is an epimorphism $\xi: \mathfrak{A}_{\mathfrak{K}} \to \mathfrak{B}_{\mathfrak{K}}$ such that $\alpha\xi = \varphi\beta$ i.e. which makes the diagram*

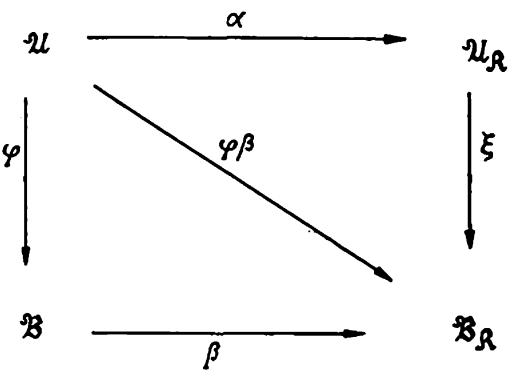

Fig. 2

commutative.

Indeed, consider the epimorphism $\varphi\beta: \mathfrak{A} \to \mathfrak{B}_{\mathfrak{K}}$ and observe that the epimorphism α is a $\mathfrak{K}$-morphism. It then follows that the required epimorphism ξ exists.

Theorem 1 has an important consequence.

Corollary 2. *$\mathfrak{K}$-replicas of isomorphic systems are isomorphic.*

Suppose that φ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ and that $\alpha\colon \mathfrak{A} \to \mathfrak{A}_{\mathfrak{B}}$ and $\beta\colon \mathfrak{B} \to \mathfrak{B}_{\mathfrak{B}}$ are $\mathfrak{K}$-morphisms. Then by Theorem 1, there are epimorphisms $\xi\colon \mathfrak{A}_{\mathfrak{B}} \to \mathfrak{B}_{\mathfrak{B}}$ and $\eta\colon \mathfrak{B}_{\mathfrak{B}} \to \mathfrak{A}_{\mathfrak{B}}$ satisfying

$$\alpha\xi = \varphi\beta, \qquad \beta\eta = \varphi^{-1}\alpha.$$

This implies that

$$\alpha\xi\eta = \alpha, \qquad \beta\eta\xi = \beta,$$

i.e. $\xi\eta$ and $\eta\xi$ are the identity mappings on $\mathfrak{A}_{\mathfrak{B}}$ and $\mathfrak{B}_{\mathfrak{B}}$, respectively. Thus, the homomorphisms ξ and η are mutually inverse and so ξ is the desired isomorphism of $\mathfrak{A}_{\mathfrak{B}}$ onto $\mathfrak{B}_{\mathfrak{B}}$. The second assertion of Corollary 2 is proved in a similar way.

Thus, if an algebraic system $\mathfrak{A}$ has a replica in some class $\mathfrak{K}$, this replica is determined up to isomorphism.

Keeping $\mathfrak{A}$ fixed and varying the class $\mathfrak{K}$, we shall in general also vary the replica $\mathfrak{A}_{\mathfrak{B}}$. Here we have a kind of "law of associativity".

Theorem 3. *If $\mathfrak{A}_{\mathfrak{B}}$ is a replica of $\mathfrak{A}$ in the class $\mathfrak{K}$ and $(\mathfrak{A}_{\mathfrak{B}})_{\mathfrak{L}}$ is a replica of $\mathfrak{A}_{\mathfrak{B}}$ in the subclass $\mathfrak{L} \subseteq \mathfrak{K}$, then the system $(\mathfrak{A}_{\mathfrak{B}})_{\mathfrak{L}}$ is a replica of $\mathfrak{A}$ in the class $\mathfrak{L}$.*

The proof is obvious.

From the definition of replica, it is evident that a $\mathfrak{K}$-replica of a system $\mathfrak{A}$ is in a certain sense the "largest" homomorphic image of $\mathfrak{A}$ in the class $\mathfrak{K}$. It may happen that a given system will have no replica in some class $\mathfrak{K}$. For example, it may be that there are no homomorphic images of the given system in $\mathfrak{K}$ at all. Sufficient conditions for the existence of a replica can be obtained directly from the results of the previous section if one uses the following connection between replicas and collections of defining formulas.

Lemma 4. *Let $\mathfrak{K}$ be a given class of algebraic systems of signature Ω and suppose that $\mathfrak{A}$ is a system of the same signature. With each element $a \in \mathfrak{A}$, we associate the symbol z_a and we let $D_a(\mathfrak{A})$ denote the collection of all atomic formulas of signature Ω in the variables z_a which are true in $\mathfrak{A}$ under the mapping $z_a \to a$ ($D_a(\mathfrak{A})$ is the atomic diagram of the system $\mathfrak{A}$). If the generating symbols z_a and the collection of formulas $D_a(\mathfrak{A})$ define a system $\mathfrak{B}$ in the class $\mathfrak{K}$, then $\mathfrak{B}$ is a $\mathfrak{K}$-replica of $\mathfrak{A}$.*

Suppose $z_a \to b_a$ is the canonical mapping of the generating symbols into the system $\mathfrak{B}$. Consider the mapping $\alpha\colon a \to b_a$ of the system $\mathfrak{A}$ into $\mathfrak{B}$. If for $a_1, \ldots, a_m$ in $\mathfrak{A}$, $P(a_1, \ldots, a_m) = \mathsf{T}$ $(P \in \Omega)$, then the formula $P(z_{a_1}, \ldots, z_{a_m})$ belongs to $D_a(\mathfrak{A})$ and hence $P(b_{a_1}, \ldots, b_{a_m}) = \mathsf{T}$.

This shows that α is a homomorphism. Since the image of $\mathfrak{A}$ in $\mathfrak{B}$ contains the elements b_a which generate the system $\mathfrak{B}$, α is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Now suppose that we are given a homomorphism $\gamma: a \to c_a$ of $\mathfrak{A}$ into a $\mathfrak{K}$-system $\mathfrak{C}$. Let us show that the relation ξ consisting of all possible pairs of the form $\langle b_a, c_a \rangle$ is a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$.

Suppose $b_a = b_{a_1}$ for a and $a_1 \in \mathfrak{A}$. Then the formula $z_a = z_{a_1}$ must be a consequence in the class $\mathfrak{K}$ of the collection of defining formulas $D_a(\mathfrak{A})$. But all the formulas of $D_a(\mathfrak{A})$ are true in $\mathfrak{C}$ under the mapping $z_a \to c_a$ since the mapping $\gamma: a \to c_a$ is a homomorphism. Therefore, the formula $z_a = z_{a_1}$ must also be true in $\mathfrak{C}$, i.e. $c_a = c_{a_1}$. We have shown that the mapping ξ is single-valued. In a similar way, it can be proved that ξ is a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$.

It is clear that the homomorphism ξ satisfies $\gamma = \alpha\xi$ and hence, α is a $\mathfrak{K}$-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Theorem 5. *Every system $\mathfrak{A}$ of signature Ω has a replica in a class $\mathfrak{K}$ of algebraic systems of signature Ω if and only if $\mathfrak{K}$ is hereditary, multiplicatively closed and contains a trivial system.*

Sufficiency follows from Theorem 4 (Section 11.2) and Lemma 4. Hence, we need only prove the *necessity* of the conditions. Thus, suppose that every system $\mathfrak{A}$ of signature Ω has a replica in the class $\mathfrak{K}$. Let $\mathfrak{A}$ be a trivial system $\mathfrak{A}_e$. The $\mathfrak{K}$-replica of $\mathfrak{A}_e$ is a homomorphic image of $\mathfrak{A}_e$. But homomorphic images of trivial systems must be trivial systems and hence, $\mathfrak{A}_e \in \mathfrak{K}$.

Let us prove that $\mathfrak{K}$ is hereditary. Let $\mathfrak{B}$ be a subsystem of the $\mathfrak{K}$-system $\mathfrak{A}$ with $\alpha: \mathfrak{B} \to \mathfrak{B}_\mathfrak{B}$ the corresponding $\mathfrak{K}$-morphism. Then for the identity embedding $\varepsilon: \mathfrak{B} \to \mathfrak{A}$, there must exist a homomorphism $\xi: \mathfrak{B}_\mathfrak{B} \to \mathfrak{A}$ such that $\varepsilon = \alpha\xi$. Since α is an epimorphism and ε is the identity mapping on $\mathfrak{B}$, the fact that $\varepsilon = \alpha\xi$ implies α and ξ are mutually inverse mappings and hence, the system $\mathfrak{B}$ is isomorphic to system $\mathfrak{B}_\mathfrak{B}$ which belongs to the class $\mathfrak{K}$.

It remains to show that $\mathfrak{K}$ is multiplicatively closed. Suppose

$$\mathfrak{A} = \prod \mathfrak{A}_i \qquad (i \in I), \quad (\mathfrak{A}_i \in \mathfrak{K}).$$

Consider the $\mathfrak{K}$-morphism

$$\alpha: \mathfrak{A} \to \mathfrak{A}_\mathfrak{A}$$

and the projections

$$\pi_i: \mathfrak{A} \to \mathfrak{A}_i \qquad (i \in I).$$

Since $\mathfrak{A}_i \in \mathfrak{K}$, there are homomorphisms

$$\xi_i: \mathfrak{A}_\mathfrak{A} \to \mathfrak{A}_i \qquad (i \in I), \tag{1}$$

which satisfy

$$\pi_i = \alpha\xi_i. \tag{2}$$

By the basic theorem on direct products (Section 2.5), to the family of homomorphisms (1), there corresponds a homomorphism $\xi\colon \mathfrak{A}_{\mathfrak{B}} \to \mathfrak{A}$ satisfying

$$\xi_i = \xi\pi_i. \tag{3}$$

From (2) and (3), it follows that $\pi_i = \alpha\xi\pi_i$, i.e. for any x in $\mathfrak{A}$,

$$x\pi_i = x(\alpha\xi)\pi_i \qquad (i \in I)$$

and hence $x = x(\alpha\xi)$.

Thus, the mapping α is an epimorphism and $\alpha\xi$ is the identity mapping. This implies that $\xi = \alpha^{-1}$ and α is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_{\mathfrak{B}}$, which belongs to the class $\mathfrak{K}$ by definition.

A class $\mathfrak{K}$ of signature Ω is called *replete (replica-complete)* if every algebraic system of signature Ω has a replica in $\mathfrak{K}$.

Corollary 6. *An arbitrary collection of generating symbols z_i $(i \in I)$ and an arbitrary collection of quasi-atomic formulas in these symbols define (in $\mathfrak{K}$) some $\mathfrak{K}$-system if and only if the class $\mathfrak{K}$ is replete.*

Sufficiency follows from Theorem 5 and Theorem 4 (Section 11.2). Necessity is a consequence of Theorem 5 and Lemma 4.

Corollary 7. *An axiomatizable class $\mathfrak{K}$ of algebraic systems is replete if and only if $\mathfrak{K}$ is a quasivariety.*

Indeed, every quasivariety is necessarily hereditary, multiplicatively closed and contains a trivial system. Conversely, if a class $\mathfrak{K}$ has these three properties and is also axiomatizable, then by Theorem 2 (Section 11.1), it is a quasivariety.

Given an arbitrary class $\mathfrak{K}$, it is not difficult to construct explicitly the minimal replete class $R\mathfrak{K}$ which contains $\mathfrak{K}$. $R\mathfrak{K}$ is called the *replete closure* of $\mathfrak{K}$.

Recall that for any class $\mathfrak{K}$, $\prod \mathfrak{K}$ symbolizes the class of all (isomorphic) copies of direct products of $\mathfrak{K}$-systems and $S\mathfrak{K}$ denotes the class of all subsystems of $\mathfrak{K}$-systems. Also, let $\mathfrak{K}_e$ be the class resulting when a trivial system is added to $\mathfrak{K}$.

Theorem 8. *For an arbitrary class of algebraic systems $\mathfrak{K}$, the class $R\mathfrak{K} = S\prod \mathfrak{K}_e$ is the minimal replete class containing $\mathfrak{K}$.*

The minimality of the class $S\prod \mathfrak{K}_e$ is obvious since a replete class contains the trivial system, the direct products of its systems and the subsystems of its systems. Therefore, we need only prove that $S\prod \mathfrak{K}_e$

is replete. Of course, this class contains the trivial system. If $\mathfrak{A}$ is a subsystem of the product $\prod \mathfrak{A}_\alpha$ ($\alpha \in I$, $\mathfrak{A}_\alpha \in \mathfrak{K}_e$), then every subsystem of $\mathfrak{A}$ is also a subsystem of this product. Therefore, the class $S\prod \mathfrak{K}_e$ is hereditary. Finally, suppose we are given some sequence $\{\mathfrak{A}_i \mid i \in I\}$ of systems in $S\prod \mathfrak{K}_e$. By hypothesis,

$$\mathfrak{A}_i \subseteq \prod \mathfrak{B}_\lambda \quad (\lambda \in K_i) \qquad (\mathfrak{B}_\lambda \in \mathfrak{K}_e;\ K_i \cap K_j = \varnothing,\ i \neq j).$$

But in that case, by Section 2.5, there exists an embedding

$$\prod \mathfrak{A}_i \subseteq \prod_i \prod_\lambda \mathfrak{B}_\lambda \cong \prod \mathfrak{B}_\nu \qquad (\nu \in \bigcup K_i),$$

from which it follows that $\prod \mathfrak{A}_i \in S\prod \mathfrak{K}_e$.

We point out the important case where (up to isomorphism) the class $\mathfrak{K}$ consists of just one system $\mathfrak{A}$. By Theorem 8, the minimal replete class containing $\mathfrak{A}$ consists of the trivial system and isomorphic copies of subsystems of direct powers of $\mathfrak{A}$.

Here is a simple example of a replete class which is not a quasivariety. Let $\mathfrak{K}_\omega$ be the class of all groups $\mathfrak{G}$ for which the intersection $\bigcap \mathfrak{G}^{(n)}$ of the successive commutator groups $\mathfrak{G}^{(0)} \supseteq \mathfrak{G}^{(1)} \supseteq \mathfrak{G}^{(2)} \supseteq \ldots$ ($\mathfrak{G}^{(0)} = \mathfrak{G}$, $\mathfrak{G}^{(n+1)} = (\mathfrak{G}^{(n)\prime})$ is the trivial group. The class $\mathfrak{K}_\omega$ is obviously hereditary, multiplicatively closed and contains the trivial group. Consequently, $\mathfrak{K}_\omega$ is replete. At the same time, it not locally closed. Indeed, as established by Ado [1] and Schmidt [63], there exist locally solvable groups coinciding with their commutator group. Therefore, the class $\mathfrak{K}_\omega$ which includes all solvable groups, does not contain all locally solvable groups.

A question that arises is: For which classes $\mathfrak{K}$ is the minimal replete class $S\prod \mathfrak{K}_e$ a quasivariety? A useful partial solution to this question is contained in the following proposition.

Corollary 9. *If a class $\mathfrak{K}$ of algebraic systems is axiomatizable, then the class $S\prod \mathfrak{K}_e$ is a quasivariety.*

It is clear that the adjoining of the trivial system to an axiomatizable class $\mathfrak{K}$ again yields an axiomatizable class $\mathfrak{K}_e$. By Corollary 5 of Section 8.3 if the class $\mathfrak{K}_e$ is axiomatizable, it follows that the class $S\prod \mathfrak{K}_e$ is universally axiomatizable.*) Now applying Corollary 7, we find that $S\prod \mathfrak{K}_e$ is a quasivariety.

For example, the class $(\mathfrak{A})$ of isomorphic copies of a finite algebraic system $\mathfrak{A}$ is axiomatizable. In view of Corollary 9, it follows in this case that the minimal replete class $S\prod (\mathfrak{A})_e$ is a quasivariety.

---

*) It is sufficient to check that any ultraproduct of $\Pi \mathfrak{K}_e$-systems is isomorphically embeddable in a suitable $\Pi \mathfrak{K}_e$-system [*Ed.*].

It was remarked above that if an epimorphism $\alpha\colon \mathfrak{A} \to \mathfrak{B}$ is a $\mathfrak{K}$-morphism for a class of systems $\mathfrak{K}$, then α is an $\mathfrak{L}$-morphism for any smaller class $\mathfrak{L}$ containing the system $\mathfrak{B}$. The converse is not true in general. Therefore, the following proposition furnishing conditions under which the converse holds, is of some interest.

Theorem 10. *If an epimorphism $\alpha\colon \mathfrak{A} \to \mathfrak{B}$ is an $\mathfrak{L}$-morphism for a class $\mathfrak{L}$, then α is a $\mathfrak{K}$-morphism for the replete closure $\mathfrak{K} = S\prod \mathfrak{L}_e$ of the class $\mathfrak{L}$.*

Let $\gamma\colon \mathfrak{A} \to \mathfrak{C}$ be a homomorphism, where

$$\mathfrak{C} = \prod_{i \in I} \mathfrak{C}_i \qquad (\mathfrak{C}_i \in \mathfrak{L}).$$

Consider the homomorphisms $\gamma\pi_i\colon \mathfrak{A} \to \mathfrak{C}_i$ where π_i is the projection of $\mathfrak{C}$ onto $\mathfrak{C}_i$. Since $\mathfrak{C}_i \in \mathfrak{L}$, there is a homomorphism

$$\xi_i\colon \mathfrak{B} \to \mathfrak{C}_i, \quad \gamma\pi_i = \alpha\xi_i \quad (i \in I). \tag{4}$$

By the basic property of direct products applied to the homomorphisms (4), there is a homomorphism $\eta\colon \mathfrak{B} \to \mathfrak{C}$ satisfying $\xi_i = \eta\pi_i$ $(i \in I)$ which with (4) yields

$$\gamma\pi_i = \alpha\eta\pi_i \qquad (i \in I),$$

i.e. $\gamma = \alpha\eta$.

We have shown that α is a $\prod \mathfrak{L}$-morphism. Analogously, suppose we are given a homomorphism $\gamma\colon \mathfrak{A} \to \mathfrak{C}$ $(\mathfrak{C} \in S\prod \mathfrak{L}_e)$. Then $\mathfrak{C}$ is isomorphic to a subsystem of a system $\mathfrak{M} \in \prod \mathfrak{L}_e$ and γ may be viewed as a homomorphism into the $\prod \mathfrak{L}_e$-system $\mathfrak{M}$. By what has been proved, α is a $\prod \mathfrak{L}_e$-morphism. Therefore, there is a homomorphism $\xi\colon \mathfrak{B} \to \mathfrak{M}$ satisfying $\gamma = \alpha\xi$, i.e. α is also an $S\prod \mathfrak{L}_e$-morphism.

Replicas in quasivarieties of algebras admit a more concrete description. We shall confine ourselves here to the following remark.

Suppose $\mathfrak{K}$ is a quasivariety of algebras of fixed signature Ω and $\mathfrak{A}$ is an algebra of that signature. Up to isomorphism, all homomorphic images of $\mathfrak{A}$ are quotient algebras of the form $\mathfrak{A}/\theta$, where θ is a congruence on $\mathfrak{A}$. Denote by S the collection of congruences θ for which $\mathfrak{A}/\theta \in \mathfrak{K}$, and let

$$\delta_\theta\colon \mathfrak{A} \to \mathfrak{A}/\theta \qquad (\theta \in S) \tag{5}$$

be the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\theta$. The homomorphisms (5) induce the homomorphism

$$\delta\colon \mathfrak{A} \to \prod \mathfrak{A}/\theta \qquad (\theta \in S), \tag{6}$$

whose kernel is a congruence $\varkappa$ equal to the intersection of all congruences $\theta \in S$. Since the factors $\mathfrak{A}/\theta$ in the product (6) belong to the quasivariety $\mathfrak{K}$, δ is a homomorphism of $\mathfrak{A}$ onto the algebra $\mathfrak{A}^\delta$, which also belongs to $\mathfrak{K}$. This implies that $\mathfrak{A}/\varkappa$ is isomorphic to $\mathfrak{A}^\delta \in \mathfrak{K}$ and hence $\mathfrak{A}/\varkappa \in \mathfrak{K}$. Thus, *among all the congruences on* $\mathfrak{A}$ *whose quotient algebras belong to the quasivariety* $\mathfrak{K}$, *there is a minimal congruence* $\varkappa$. Let us show that $\mathfrak{A}/\varkappa$ is a $\mathfrak{K}$-replica of $\mathfrak{A}$ and that the canonical homomorphism $\delta_\varkappa\colon \mathfrak{A} \to \mathfrak{A}/\varkappa$ is a $\mathfrak{K}$-morphism. Consider some homomorphism $\gamma\colon \mathfrak{A} \to \mathfrak{C}$ of the algebra $\mathfrak{A}$ onto an arbitrary $\mathfrak{K}$-algebra $\mathfrak{C}$. This homomorphism induces the isomorphism $\varphi\colon \mathfrak{A}/\sigma \to \mathfrak{C}$, where σ is the kernel congruence of γ. From $\mathfrak{C} \in \mathfrak{K}$, we have $\mathfrak{A}/\sigma \in \mathfrak{K}$, and hence, $\varkappa \subseteq \sigma$. Letting ϱ denote the canonical homomorphism of $\mathfrak{A}/\varkappa$ onto $\mathfrak{A}/\sigma$, we have $\gamma = \delta_\varkappa(\varrho\varphi)$, as required.

The minimal congruence $\varkappa$ above is called the $\mathfrak{K}$-*congruence* on $\mathfrak{A}$. It depends on the choice of quasivariety $\mathfrak{K}$. For example, if the algebra $\mathfrak{A}$ is simple and $\mathfrak{A} \notin \mathfrak{K}$, then $\varkappa$ is the universal congruence. If $\mathfrak{A} \in \mathfrak{K}$, then $\varkappa$ is the trivial congruence.

A congruence $\varkappa$ on an algebra $\mathfrak{A}$ is called *quasiverbal* if there exists a quasivariety $\mathfrak{K}$ for which the canonical mapping $\mathfrak{A} \to \mathfrak{A}/\varkappa$ is a $\mathfrak{K}$-morphism. The congruence $\varkappa$ is called *verbal* if there exists a variety $\mathfrak{K}$ with this property. The notion of a quasiverbal or verbal quotient system of an arbitrary system is defined in the same way.

We now give several examples of $\mathfrak{K}$-congruences. Let $\mathfrak{K}$ be the quasivariety of cancellation semigroups. From the preceding, for any semigroup $\mathfrak{A}$ there is a minimal congruence $\varkappa$ such that $\mathfrak{A}/\varkappa$ is a cancellation semigroup.

In Section 8.3, it was shown that the class of semigroups embeddable in groups is a quasivariety. Therefore, there is a minimal congruence on every semigroup $\mathfrak{A}$ which yields a quotient semigroup embeddable in a group.

Explicit constructions of $\mathfrak{K}$-congruences are often found in group theory. For instance, if $\mathfrak{K}$ is the variety of abelian groups, then a $\mathfrak{K}$- replica of a nonabelian group is its quotient group modulo the commutator group. Taking $\mathfrak{K}$ to be the class of all metabelian groups, we can easily show that the $\mathfrak{K}$-replica of an arbitrary group $\mathfrak{A}$ will be the quotient group $\mathfrak{A}/\mathfrak{A}^3$ (modulo the second member of the lower central series for $\mathfrak{A}$).

Several additional properties of verbal congruences will be given below in Section 14.2.

Closely related to the notion of $\mathfrak{K}$-congruence is the still little-explored concept of attainability of a class (Tamura [69]). Suppose we are given a quasivariety $\mathfrak{K}$ of signature Ω. Let $\mathfrak{A}$ be an algebra of the same signature. The algebra $\mathfrak{A}$ is called $\mathfrak{K}$-*indecomposable* if the $\mathfrak{K}$-congruence $\varkappa$ on $\mathfrak{A}$ is the universal congruence.

Suppose that the $\mathfrak{K}$-congruence $\varkappa$ is not universal. Then the algebra $\mathfrak{A}$ splits into $\varkappa$-equivalence classes of which certain ones may be subalgebras of $\mathfrak{A}$. If all these subalgebras are $\mathfrak{K}$-indecomposable, then the class $\mathfrak{K}$ is said to be *attainable* in $\mathfrak{A}$ and the algebra $\mathfrak{A}$ is said to be $\mathfrak{K}$-*attainable*. A class of algebras $\mathfrak{L}$ is called $\mathfrak{K}$-*attainable* if it consists of merely $\mathfrak{K}$-attainable algebras. If the class $\mathfrak{L}$ is $\mathfrak{K}$-attainable, then the quasivariety $\mathfrak{K}$ is also said to be *attainable in the class* $\mathfrak{L}$.

Tamura [69] showed that the only nontrivial variety of semigroups attainable in all semigroups is the variety of commutative idempotent semigroups described by the identities

$$x \cdot yz = xy \cdot z, \quad x^2 = x, \quad xy = yx.$$

He also showed there are no nontrivial varieties of groups attainable in all groups. This holds also for abelian groups. The question of finding varieties of groupoids, rings and lattices attainable in the classes of all groupoids, all rings or all lattices is left open in the above paper by Tamura. The question also remains open as to which varieties and quasivarieties of semigroups with several distinguished elements are attainable in the class of all such semigroups.*)

Problems and Complements

1. The quasiprimitive and replete closures of any axiomatizable class $\mathfrak{K}$ of algebraic systems all coincide.

2. Suppose that $\mathfrak{G}$ is a semigroup presented by a finite or countable collection of generators $z_1, z_2, \ldots$ and some collection $\mathfrak{S}$ of defining relations. Substitution of the words

$$z_i = a_1 a_2 a_1^{i+1} a_2^{i+1} \quad (i = 1, 2, \ldots) \tag{1}$$

for the symbols z_i in the relations belonging to $\mathfrak{S}$ yields a set of defining relations for a semigroup $\mathfrak{G}_0$ with generators a_1 and a_2. Formulas (1) define an isomorphic embedding of $\mathfrak{G}$ into $\mathfrak{G}_0$ and $a_1 x \neq x$ for all x in $\mathfrak{G}_0$ (Hall [17] and also Mal'cev [47]).

12. Free Systems and Free Compositions

Free compositions (products) and free systems play a significant role in the theory of classes. This section will present just the definitions and basic properties of free systems and free compositions.

12.1 Free Compositions. Let $\mathfrak{K}$ be a class of algebraic systems of signature Ω and suppose we have a sequence $\{\mathfrak{A}_i \mid i \in I\}$ of systems of this signature but not necessarily in the class $\mathfrak{K}$. A system $\mathfrak{A}$ of signature Ω

*) One condition for the attainability of varieties and quasivarieties of arbitrary algebraic systems was given by Mal'cev in [49] [*Ed.*].

Suppose that the $\mathfrak{K}$-congruence $\varkappa$ is not universal. Then the algebra $\mathfrak{A}$ splits into $\varkappa$-equivalence classes of which certain ones may be subalgebras of $\mathfrak{A}$. If all these subalgebras are $\mathfrak{K}$-indecomposable, then the class $\mathfrak{K}$ is said to be *attainable* in $\mathfrak{A}$ and the algebra $\mathfrak{A}$ is said to be $\mathfrak{K}$-*attainable*. A class of algebras $\mathfrak{L}$ is called $\mathfrak{K}$-*attainable* if it consists of merely $\mathfrak{K}$-attainable algebras. If the class $\mathfrak{L}$ is $\mathfrak{K}$-attainable, then the quasivariety $\mathfrak{K}$ is also said to be *attainable in the class* $\mathfrak{L}$.

Tamura [69] showed that the only nontrivial variety of semigroups attainable in all semigroups is the variety of commutative idempotent semigroups described by the identities

$$x \cdot yz = xy \cdot z, \quad x^2 = x, \quad xy = yx.$$

He also showed there are no nontrivial varieties of groups attainable in all groups. This holds also for abelian groups. The question of finding varieties of groupoids, rings and lattices attainable in the classes of all groupoids, all rings or all lattices is left open in the above paper by Tamura. The question also remains open as to which varieties and quasivarieties of semigroups with several distinguished elements are attainable in the class of all such semigroups.*)

Problems and Complements

1. The quasiprimitive and replete closures of any axiomatizable class $\mathfrak{K}$ of algebraic systems all coincide.

2. Suppose that $\mathfrak{G}$ is a semigroup presented by a finite or countable collection of generators $z_1, z_2, \ldots$ and some collection $\mathfrak{S}$ of defining relations. Substitution of the words

$$z_i = a_1 a_2 a_1^{i+1} a_2^{i+1} \quad (i = 1, 2, \ldots) \tag{1}$$

for the symbols z_i in the relations belonging to $\mathfrak{S}$ yields a set of defining relations for a semigroup $\mathfrak{G}_0$ with generators a_1 and a_2. Formulas (1) define an isomorphic embedding of $\mathfrak{G}$ into $\mathfrak{G}_0$ and $a_1 x \neq x$ for all x in $\mathfrak{G}_0$ (Hall [17] and also Mal'cev [47]).

12. Free Systems and Free Compositions

Free compositions (products) and free systems play a significant role in the theory of classes. This section will present just the definitions and basic properties of free systems and free compositions.

12.1 Free Compositions. Let $\mathfrak{K}$ be a class of algebraic systems of signature Ω and suppose we have a sequence $\{\mathfrak{A}_i \mid i \in I\}$ of systems of this signature but not necessarily in the class $\mathfrak{K}$. A system $\mathfrak{A}$ of signature Ω

*) One condition for the attainability of varieties and quasivarieties of arbitrary algebraic systems was given by Mal'cev in [49] [*Ed.*].

is called the *free composition* of the systems $\mathfrak{A}_i$ in the class $\mathfrak{K}$ (or their $\mathfrak{K}$-*composition*) if it satisfies the following conditions:

1) $\mathfrak{A} \in \mathfrak{K}$;

2) there exists a family of homomorphisms

$$\alpha_i\colon \mathfrak{A}_i \to \mathfrak{A} \qquad (i \in I),$$

such that

2a) the collection $\bigcup \mathfrak{A}_i^{\alpha_i}$ of images of the $\mathfrak{A}_i$ in $\mathfrak{A}$ generates $\mathfrak{A}$;

2b) to every collection of homomorphisms

$$\gamma_i\colon \mathfrak{A}_i \to \mathfrak{C}$$

of the $\mathfrak{A}_i$ into an arbitrary $\mathfrak{K}$-system $\mathfrak{C}$, there exists a homomorphism $\xi\colon \mathfrak{A} \to \mathfrak{C}$ satisfying

$$\gamma_i = \alpha_i \xi \qquad (i \in I).$$

The assertion that $\mathfrak{A}$ is the $\mathfrak{K}$-composition of systems $\mathfrak{A}_i$ $(i \in I)$ is denoted by

$$\mathfrak{A} = \prod\nolimits^*_{\mathfrak{K}} \mathfrak{A}_i \qquad (i \in I). \tag{1}$$

In the simplest case, where $I = \{0, 1, \ldots, n-1\}$ and $\mathfrak{A}_0 = \mathfrak{B}$, $\mathfrak{A}_1 = \mathfrak{C}, \ldots, \mathfrak{A}_{n-1} = \mathfrak{D}$, we shall write instead of (1)

$$\mathfrak{A} = \mathfrak{B} \overset{*}{\underset{\mathfrak{K}}{}} \mathfrak{C} \overset{*}{\underset{\mathfrak{K}}{}} \ldots \overset{*}{\underset{\mathfrak{K}}{}} \mathfrak{D}, \tag{2}$$

and the symbol $\mathfrak{K}$ in (1) and (2) will be omitted if the class $\mathfrak{K}$ has been fixed beforehand in some way. If all the systems $\mathfrak{A}_i$ are the same system $\mathfrak{B}$, then the composition (1) is called the $\mathfrak{K}$-*free I-th power* of $\mathfrak{B}$ and instead of (1) we shall write

$$\mathfrak{A} = \mathfrak{B}^{*I}(\mathfrak{K}).$$

The systems $\mathfrak{A}_i$ in the $\mathfrak{K}$-composition (1) are called its *free factors*, and the homomorphisms α_i are called the *composition* or *canonical homomorphisms* of the free factors into their $\mathfrak{K}$-composition.

Suppose that I consists of only one element s. Comparing the definition of the $\mathfrak{K}$-composition $\prod^*_{\mathfrak{K}} \mathfrak{A}_i$ $(i \in \{s\})$ with the definition of the $\mathfrak{K}$-replica $\mathfrak{A}_s$ (Section 11.3), we see that $\prod^*_{\mathfrak{K}} \mathfrak{A}_i$ $(i = s)$ is simply the replica $\mathfrak{A}_s$ in the class $\mathfrak{K}$. Therefore, we may view the notion of $\mathfrak{K}$-composition as a generalization of the notion of $\mathfrak{K}$-replica. Just as a $\mathfrak{K}$-replica is defined by a collection of defining relations, the $\mathfrak{K}$-composition (1) can also be presented by a natural set of defining relations. From this presentation, we shall easily obtain a number of properties of free compositions.

With each element $a \in \mathfrak{A}_i$, we associate a special symbol z_a^i and we denote by $D_a(\mathfrak{A}_i)$ the *atomic diagram of the system* $\mathfrak{A}_i$ expressed through the symbols z_a^i, i.e. the collection of formulas of the form

$$P(z_a^i, \ldots, z_c^i), \quad z_a^i = F(z_b^i, \ldots, z_c^i) \quad (P, F \in \Omega),$$

which are true in $\mathfrak{A}_i$ under the mapping $z_a^i \to a$. Note that due to the superscripts on the symbols z_a^i, the diagrams of the factors $\mathfrak{A}_i$ and $\mathfrak{A}_j$ will for $i \neq j$ consist of different formulas even when $\mathfrak{A}_i = \mathfrak{A}_j$.

Theorem 1. *A system* $\mathfrak{A}$ *presented in the class* $\mathfrak{K}$ *by generating symbols* $z_a^i (i \in I,\ a \in \mathfrak{A}_i)$ *and defining relations* $\bigcup D_a(\mathfrak{A}_i)$ $(i \in I)$ *is the free composition of the* $\mathfrak{A}_i (i \in I)$ *in* $\mathfrak{K}$. *Moreover, the canonical mapping* $\varphi: z_a^i \to \mathfrak{A}$ *of the generating symbols into* $\mathfrak{A}$ *induces composition homomorphisms* $\alpha_i: a \to z_a^i \varphi$ *of the systems* $\mathfrak{A}_i$ *into their composition* $\mathfrak{A}$.

To prove this, one need only compare the definition of the free $\mathfrak{K}$-composition of systems $\mathfrak{A}_i$ with the definition of a system presented in the class $\mathfrak{K}$ by generating symbols z_a^i and defining relations $\bigcup D_a(\mathfrak{A}_i)$.

The same considerations show in fact that the following slightly more general statement is true.

Theorem 1a. *Suppose that systems* $\mathfrak{A}_i$ $(i \in I)$ *are presented in the class* $\mathfrak{K}$ *by pairwise disjoint generating sets* Z_i *and certain collections* T_i *of defining relations. Then the system presented in* $\mathfrak{K}$ *by* $\bigcup Z_i$, *the union of the generating sets, and* $\bigcup T_i$, *the union of the defining relations, is the free composition of the* $\mathfrak{A}_i$ *in* $\mathfrak{K}$.

From Theorem 1, we obtain the next result.

Corollary 2. *Let*

$$\mathfrak{A} = \prod * \mathfrak{A}_i \quad (i \in I), \qquad \mathfrak{B} = \prod * \mathfrak{B}_j \quad (j \in J),$$

be given $\mathfrak{K}$*-compositions with*

$$\alpha_i: \mathfrak{A}_i \to \mathfrak{A}, \qquad \beta_j: \mathfrak{B}_j \to \mathfrak{B}$$

their corresponding composition homomorphisms. Suppose φ *is a one-to-one mapping of* I *onto* J *and*

$$\chi_i: \mathfrak{A}_i \to \mathfrak{B}_{i\varphi} \qquad (i \in I)$$

are homomorphisms of the factors of the first composition into the corresponding factors of the second. Then there exists a homomorphism ξ *of* $\mathfrak{A}$ *into* $\mathfrak{B}$ *satisfying*

$$\alpha_i \xi = \chi_i \beta_{i\varphi} \qquad (i \in I).$$

Moreover, if the χ_i are homomorphisms of $\mathfrak{A}_i$ onto $\mathfrak{B}_{i\varphi}$ then ξ will be a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$; if the χ_i are isomorphisms of $\mathfrak{A}_i$ onto $\mathfrak{B}_{i\varphi}$, then ξ will be an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

By Theorem 1, the system $\mathfrak{A}$ is presented in the class $\mathfrak{K}$ by the generating symbols $z_a{}^i (a \in \mathfrak{A}_i,\ i \in I)$ and defining relations $\cup D_a(\mathfrak{A}_i)$. Since χ_i and β_j are homomorphisms, the formulas in $D_a(\mathfrak{A}_i)$ are true in $\mathfrak{B}$ for values of $z_a{}^i \to a\chi_i\beta_{i\varphi} (a \in \mathfrak{A}_i)$. From the definition of defining relations, it follows that there exists a homomorphism $\xi\colon \mathfrak{A} \to \mathfrak{B}$ for which $a\alpha_i\xi = a\chi_i\beta_{i\varphi}$ i.e. $\alpha_i\xi = \chi_i\beta_{i\varphi}$. If the homomorphisms χ_i are epimorphisms, then $\mathfrak{A}\xi$ contains all the systems $\mathfrak{B}_j\beta_j$. The collection of systems $\mathfrak{B}_j\beta_j$ generate $\mathfrak{B}$ and hence $\mathfrak{A}\xi = \mathfrak{B}$. If the epimorphisms χ_i are isomorphisms, then we may exchange the roles of $\mathfrak{A}$ and $\mathfrak{B}$ to obtain the inverse epimorphism $\xi^{-1}\colon \mathfrak{B} \to \mathfrak{A}$.

From the examples below, it will be clear that in contrast to direct and filtered products, one can not say here that if the $\varkappa_i$ are isomorphisms of $\mathfrak{A}_i$ into $\mathfrak{B}_{i\varphi}$, then ξ will also be an isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Theorem 3 (Associativity of Free Compositions). *Let $\mathfrak{K}$ be a class of algebraic systems of signature Ω and let $\{\mathfrak{A}_i \mid i \in I\}$ be a sequence of systems of signature Ω. Partition I into arbitrary pairwise disjoint nonempty subsets I_μ $(\mu \in M)$. Then*

$$\prod_{i\in I}{}^{*}\, \mathfrak{A}_i \cong \prod_{\mu\in M}{}^{*} \left(\prod_{i\in I_\mu}{}^{*}\, \mathfrak{A}_i\right).$$

Indeed, suppose

$$\mathfrak{B}_\mu = \prod{}^{*}\, \mathfrak{A}_i \quad (i \in I_\mu), \qquad \mathfrak{B} = \prod{}^{*}\, \mathfrak{B}_\mu \quad (\mu \in M)$$

and

$$\beta_{i\mu}\colon \mathfrak{A}_i \to \mathfrak{B}_\mu \quad (i \in I_\mu), \qquad \beta_\mu\colon \mathfrak{B}_\mu \to \mathfrak{B} \quad (\mu \in M)$$

are the corresponding composition homomorphisms. Let us show that $\mathfrak{B}$ is the $\mathfrak{K}$-composition of the systems $\mathfrak{A}_i$ $(i \in I)$ with composition homomorphisms $\alpha_i = \beta_{i\mu}\beta_\mu\colon \mathfrak{A}_i \to \mathfrak{B}$. Suppose we are given arbitrary homomorphisms $\gamma_i\colon \mathfrak{A}_i \to \mathfrak{C}$ into an arbitrary $\mathfrak{K}$-system $\mathfrak{C}$. Since $\mathfrak{B}_\mu$ is the $\mathfrak{K}$-composition of the $\mathfrak{A}_i$ $(i \in I_\mu)$, there must be a homomorphism

$$\xi_\mu\colon \mathfrak{B}_\mu \to \mathfrak{C} \quad (\mu \in M),$$

satisfying

$$\gamma_i = \beta_{i\mu}\xi_\mu \quad (i \in I_\mu). \tag{3}$$

Since $\mathfrak{B}$ is the $\mathfrak{K}$-composition of the $\mathfrak{B}_\mu$ $(\mu \in M)$, there must exist a homomorphism $\xi\colon \mathfrak{B} \to \mathfrak{C}$ such that

$$\xi_\mu = \beta_\mu\xi \quad (\mu \in M). \tag{4}$$

From (3) and (4), we obtain the equations

$$\gamma_i = \beta_{i\mu}\beta_\mu\xi \qquad (i \in I_\mu),$$

which show that $\mathfrak{B}$ is the $\mathfrak{K}$-composition of the $\mathfrak{A}_i$ $(i \in I)$. By Corollary 2, $\mathfrak{K}$-compositions are uniquely determined up to isomorphism.

Recall that a class $\mathfrak{K}$ of signature Ω is replete if every system $\mathfrak{A}$ of signature Ω has a certain replica $\mathfrak{A}_\mathfrak{B}$ in $\mathfrak{K}$. We remarked above that the concept of $\mathfrak{K}$-composition is a generalization of the notion of replica. The simplest connection between these two ideas is given by the next theorem.

Theorem 4. *Let $\mathfrak{K}$ be a replete class of signature Ω. Then for any sequence $\{\mathfrak{A}_i \mid i \in I\}$ of systems of signature Ω,*

$$\prod\nolimits^*_\mathfrak{B} \mathfrak{A}_i \cong \prod\nolimits^*_\mathfrak{B} (\mathfrak{A}_i)_\mathfrak{B} \qquad (i \in I) \tag{5}$$

and for any replete subclass $\mathfrak{L} \subseteq \mathfrak{K}$,

$$\prod\nolimits^*_\mathfrak{L} \mathfrak{A}_i \cong (\prod\nolimits^*_\mathfrak{B} \mathfrak{A}_i)_\mathfrak{L}. \tag{6}$$

Let us prove the first assertion. Suppose

$$\mathfrak{B} = \prod\nolimits^*_\mathfrak{B} (\mathfrak{A}_i)_\mathfrak{B} \qquad (i \in I)$$

and

$$\beta_i\colon\ (\mathfrak{A}_i)_\mathfrak{B} \to \mathfrak{B},$$

$$\varkappa_i\colon\ \mathfrak{A}_i \to (\mathfrak{A}_i)_\mathfrak{B}$$

are the corresponding composition homomorphisms and replica $\mathfrak{K}$-morphisms. We wish to show that $\mathfrak{B}$ is the $\mathfrak{K}$-composition of the systems $\mathfrak{A}_i (i \in I)$ associated with the composition homomorphisms

$$\alpha_i = \varkappa_i\beta_i\colon\ \mathfrak{A}_i \to \mathfrak{B}.$$

First of all, it is clear that the collections $\mathfrak{A}_i^{\varkappa_i} = (\mathfrak{A}_i)^{\beta_i}_\mathfrak{B}$ generate $\mathfrak{B}$. Next, suppose we are given homomorphisms

$$\gamma_i\colon\ \mathfrak{A}_i \to \mathfrak{C}$$

into some $\mathfrak{K}$-system $\mathfrak{C}$. By the basic property of $\mathfrak{K}$-replicas, this implies that there are homomorphisms $\xi_i\colon\ (\mathfrak{A}_i)_\mathfrak{B} \to \mathfrak{C}$ satisfying

$$\gamma_i = \varkappa_i\xi_i. \tag{7}$$

Since the system $\mathfrak{B}$ is the $\mathfrak{K}$-composition of the systems $(\mathfrak{A}_i)_{\mathfrak{B}}$, there exists a homomorphism $\xi\colon \mathfrak{B} \to \mathfrak{C}$ such that $\xi_i = \beta_i \xi$. Comparing these equations with (7), we obtain

$$\gamma_i = (\varkappa_i \beta_i)\,\xi \qquad (i \in I),$$

thus showing that $\mathfrak{B}$ is the $\mathfrak{K}$-composition of the $\mathfrak{A}_i$.

The second statement in the theorem is proved analogously. $\mathfrak{K}$-composition may be viewed as an operation which can be applied to any collection of systems of signature Ω. If the systems are viewed only up to isomorphism, i.e. instead of the systems one considers their isoclasses (Section 2.2), then the above theorems show that the operation of $\mathfrak{K}$-composition will be well-defined, commutative and associative. For instance, if we let $U_\Omega{}^{\mathfrak{m}}$ denote the collection of isoclasses of systems of finite signature Ω whose cardinality does not exceed the infinite cardinal $\mathfrak{m}$, then the collection $U_\Omega{}^{\mathfrak{m}}$ will simply be a commutative semigroup $\langle U_\Omega{}^{\mathfrak{m}}, *\rangle$ under $\mathfrak{K}$-composition of pairs of systems.

Formally, the operation of $\mathfrak{K}$-composition is defined for any systems of signature Ω including those which do not belong to the class $\mathfrak{K}$. However, the isomorphism (5) shows that the result of the composition depends not on the factors $\mathfrak{A}_i$ themselves, but only on their $\mathfrak{K}$-replicas. Hence every thing reduces to the consideration of $\mathfrak{K}$-compositions of $\mathfrak{K}$-systems.

The composition homomorphisms $\alpha_i\colon \mathfrak{A}_i \to \mathfrak{A}$ map the factors $\mathfrak{A}_i$ onto certain subsystems $\mathfrak{A}_i^{\alpha_i}$ of the composition $\mathfrak{A}$. Under what conditions will each homomorphism α_i be an isomorphism? Clearly, for this it is necessary that $\mathfrak{A}_i$ belong to $S\mathfrak{K}$. However, there are examples which show that for several important classes of systems, the composition homomorphisms need not be isomorphisms even in those cases where all the factors belong to $\mathfrak{K}$.

Nevertheless, from the remarks below it will be seen that under rather natural conditions, the composition homomorphisms are isomorphisms.

We shall say that the systems $\mathfrak{A}_i$ $(i \in I)$ are *compatibly embeddable* in a $\mathfrak{K}$-system if for a suitable $\mathfrak{K}$-system $\mathfrak{A}$ there is a collection of isomorphisms

$$\alpha_i\colon \mathfrak{A}_i \to \mathfrak{A} \qquad (i \in I).$$

We shall say that the $\mathfrak{K}$-composition of a collection of systems $\mathfrak{A}_i (i \in I)$ is *injective* if all its composition homomorphisms are isomorphisms.

These two definitions have a connection.

Corollary 5. *A $\mathfrak{K}$-composition of a collection of systems $\mathfrak{A}_i$ $(i \in I)$ is injective if and only if the $\mathfrak{A}_i (i \in I)$ are compatibly embeddable in some suitable-$\mathfrak{K}$-system.*

From the Compactness Theorem (Section 8.3), it follows that if any pair of systems of an axiomatizable class $\mathfrak{K}$ are compatibly embeddable in a $\mathfrak{K}$-system, then any collection of $\mathfrak{K}$-systems is compatibly embeddable in some $\mathfrak{K}$-system. Since an arbitrary quasivariety $\mathfrak{K}$ and the class of all its nontrivial systems are axiomatizable classes, Corollary 5 mplies the following proposition.

Theorem 6. *If any pair of systems (nontrivial systems) of a quasivariety $\mathfrak{K}$ are compatibly embeddable in a corresponding $\mathfrak{K}$-system, then the $\mathfrak{K}$-composition of an arbitrary collection of $\mathfrak{K}$-systems (nontrivial $\mathfrak{K}$-systems) is injective.*

Besides these general remarks, we shall point out a more specific property of injective compositions.

Theorem 7. *If the factor $\mathfrak{A}_s$ in the $\mathfrak{K}$-composition $\mathfrak{A} = \prod * \mathfrak{A}_i$ $(i \in I)$ belongs to $\mathfrak{K}$ and every other factor $\mathfrak{A}_i$ can be homomorphically mapped into $\mathfrak{A}_s$, then the composition homomorphism $\alpha_s: \mathfrak{A}_s \to \mathfrak{A}$ is an isomorphism.*

Taking the identity mapping

$$\gamma_s: \mathfrak{A}_s \to \mathfrak{A}_s$$

with the homomorphisms (which exist by hypothesis)

$$\gamma_i: \mathfrak{A}_i \to \mathfrak{A}_s$$

and noting that $\mathfrak{A}_s \in \mathfrak{K}$, we see that there exists a homomorphism $\xi: \mathfrak{A} \to \mathfrak{A}_s$, such that $\gamma_s = \alpha_s \xi$. Since γ_s is the identity mapping, α_s must be an isomorphism of $\mathfrak{A}_s$ onto $\mathfrak{A}_s \alpha_s$.

In particular, the conditions of Theorem 7 certainly hold if all the factors are equal to the same $\mathfrak{K}$-system. Thus, *the composition homomorphisms for any $\mathfrak{K}$-power of an arbitrary $\mathfrak{K}$-system are isomorphisms.*

Another important special case involves those classes $\mathfrak{K}$ whose systems each contain a trivial subsystem. In such classes, any system can be homomorphically mapped into any other system (and onto its trivial subsystem). Thus, the hypotheses of Theorem 7 hold in compositions of any $\mathfrak{K}$-systems. Examples of such classes are groups, loops, rings, etc.

An algebraic system $\mathfrak{A}$ is called the $\mathfrak{K}$-*free product* of its subsystems $\mathfrak{A}_i (i \in I)$ if $\mathfrak{A}$ is a $\mathfrak{K}$-composition of the sequence $\mathfrak{A}_i (i \in I)$ for which the identity mappings $\mathfrak{A}_i \to \mathfrak{A}_i$ serve as composition homomorphisms. In other words, $\mathfrak{A}$ is the $\mathfrak{K}$-free product of its subsystems $\mathfrak{A}_i (i \in I)$ if three conditions are satisfied: 1) $\mathfrak{A} \in \mathfrak{K}$; 2) the collection of all elements

of the $\mathfrak{A}_i$ generates $\mathfrak{A}$; 3) whenever $\alpha_i (i \in I)$ is a given arbitrary homomorphism of $\mathfrak{A}_i$ into some $\mathfrak{K}$-system $\mathfrak{C}$, then the entire collection of homomorphisms α_i can be compatibly extended to a suitable homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$.

Suppose we are given a replete class $\mathfrak{K}$ and a sequence of $\mathfrak{K}$-systems $\{\mathfrak{B}_i \mid i \in I\}$. Does there exist a system $\mathfrak{A}$ which is the $\mathfrak{K}$-free product of a sequence of its subsystems $\{\mathfrak{A}_i \mid i \in \}$ each of which is respectively isomorphic to a given system $\mathfrak{B}_i$? Theorem 7 gives an affirmative answer to this question when any of the given systems can be homomorphically mapped into any other system.

We also point out an immediate consequence of Theorem 3.

Corollary 8. *Let $\mathfrak{K}$ be a replete class. Suppose the system $\mathfrak{A}$ is the $\mathfrak{K}$-free product of its subsystems $\mathfrak{A}_i (i \in I)$, $I = \bigcup I_\mu (\mu \in M)$ is a partition of I into disjoint subcollections and $\mathfrak{B}_\mu$ is the subsystem generated in $\mathfrak{A}$ by the elements of $\mathfrak{A}_i (i \in I_\mu)$. Then $\mathfrak{A}$ is the $\mathfrak{K}$-free product of the subsystems $\mathfrak{B}_\mu (\mu \in M)$.*

In a more detailed study of free products, it is natural to ask: If $\mathfrak{A}$ is the $\mathfrak{K}$-free product of its subsystems $\mathfrak{A}_i$ $(i \in I)$, then what is the nature of the intersections $\mathfrak{A}_i \cap \mathfrak{A}_j$ $(i \neq j)$? In view of Corollary 8, we need only consider two factors.

Theorem 9. *If a system $\mathfrak{A}$ is the $\mathfrak{K}$-free product of its subsystems $\mathfrak{A}_1$ and $\mathfrak{A}_2$, then for any homomorphism $\alpha: \mathfrak{A}_1 \to \mathfrak{A}_2$ we have $\mathfrak{A}_1 \cap \mathfrak{A}_2 \subseteq \mathfrak{A}_1^\alpha$.*

The homomorphism $\alpha: \mathfrak{A}_1 \to \mathfrak{A}_2$ and the identity mapping $\beta: \mathfrak{A}_2 \to \mathfrak{A}_2$ may be viewed as homomorphisms of the free factors into the $\mathfrak{K}$-algebra $\mathfrak{A}$. According to the definition of free product, the above homomorphisms can be extended to a homomorphism $\xi: \mathfrak{A} \to \mathfrak{A}_2$. Since $\mathfrak{A}_1 \cap \mathfrak{A}_2 \subseteq \mathfrak{A}_2$ and $\xi = \beta$ on $\mathfrak{A}_2$, we have $(\mathfrak{A}_1 \cap \mathfrak{A}_2)^\xi = \mathfrak{A}_1 \cap \mathfrak{A}_2$. On the other hand, $(\mathfrak{A}_1 \cap \mathfrak{A}_2)^\xi = (\mathfrak{A}_1 \cap \mathfrak{A}_2)^\alpha \subseteq \mathfrak{A}_1^\alpha$ and hence $\mathfrak{A}_1 \cap \mathfrak{A}_2 \subseteq \mathfrak{A}_1^\alpha$.

For example, *if $\mathfrak{A} \subseteq \mathfrak{C}$, $\mathfrak{B} \subseteq \mathfrak{C}$, $\mathfrak{C} = \mathfrak{A} * \mathfrak{B}$ and the intersection $\mathfrak{A} \cap \mathfrak{B}$ contains a trivial subsystem, then the intersection coincides with this subsystem.* In particular, every system in classes of loops, groups, etc. contains a unique trivial subsystem. Therefore, in a free product of loops or groups, a pair of free factors interesect only in the trivial subalgebra.

Let us look at the free products in the class of semigroups. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two disjoint semigroups. The elements of both semigroups will be viewed as symbols in an alphabet $\mathscr{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$. Denote by $\mathfrak{A}$ the collection of those nonempty words in the alphabet $\mathscr{A}$ in which no two adjacent symbols belong to the same semigroup. Introduce a multiplication of words $c_1 c_2 \dots c_m$ and $d_1 d_2 \dots d_n$ in $\mathfrak{A}$ by the concatenation

$$(c_1 \dots c_m) \cdot (d_1 \dots d_n) = c_1 \dots c_m d_1 \dots d_n \tag{8}$$

if c_m and d_1 belong to distinct semigroups, and by

$$(c_1 \dots c_m) \cdot (d_1 \dots d_n) = c_1 \dots (c_m d_1) \dots d_n \tag{9}$$

if $c_m, d_1 \in \mathfrak{A}_i$, where $(c_m d_1)$ denotes that element of the semigroup $\mathfrak{A}_i$ which results on multiplying c_m and d_1 in $\mathfrak{A}_i$.

It is easy to verify that the groupoid thus obtained is associative. It contains the elements of each of the semigroups $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (as words of length 1), and the multiplication of elements of these semigroups in the semigroup $\mathfrak{A}$ according to (9) coincides with the multiplication laws in the semigroups $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Consequently, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are subsemigroups of $\mathfrak{A}$.

By (8), each word in $\mathfrak{A}$ is a product of its letters, i.e. each element in $\mathfrak{A}$ is the product of corresponding elements in the subsemigroups $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Thus, $\mathfrak{A}$ is generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

To prove that $\mathfrak{A}$ is the free product of its subsemigroups $\mathfrak{A}_1$ and $\mathfrak{A}_2$, it remains to verify the condition on extending homomorphisms. Suppose $\alpha_1 \colon \mathfrak{A}_1 \to \mathfrak{G}$ and $\alpha_2 \colon \mathfrak{A}_2 \to \mathfrak{G}$ are homomorphisms into a semigroup, $\mathfrak{G}$. With each word $\mathfrak{a} = c_1 c_2 \dots c_m \in \mathfrak{A}$, we associate its value

$$||\mathfrak{a}|| = c_1^{\beta_1} c_2^{\beta_2} \dots c_m^{\beta_m} \in \mathfrak{G},$$

where $\beta_i = \alpha_1$ if $c_i \in \mathfrak{A}_1$ and $\beta_i = \alpha_2$ if $c_i \in \mathfrak{A}_2$. From (8) and (9), it is immediately clear that

$$||\mathfrak{a} \cdot \mathfrak{b}|| = ||\mathfrak{a}|| \cdot ||\mathfrak{b}|| \qquad (\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}),$$

and

$$a \in \mathfrak{A}_i \to ||\mathfrak{a}|| = a^{\alpha_i} \qquad (i = 1, 2).$$

Consequently, the mapping $\xi \colon \mathfrak{a} \to ||\mathfrak{a}||$ is the desired homomorphism of $\mathfrak{A}$ into $\mathfrak{G}$ which extends the homomorphisms α_1 and α_2.

For example, consider the structure of the free composition (in the class $\mathfrak{K}$ of all semigroups) of two trivial semigroups. Let a be the unique element of the semigroup $\mathfrak{A}$ and b the unique element of the semigroup $\mathfrak{B}$ and suppose $a \neq b$. According to the above, the $\mathfrak{K}$-free product of $\mathfrak{A}$ and $\mathfrak{B}$ can be viewed as the semigroup $\mathfrak{L}$ consisting of the words $a, b, ab, ba, aba, \dots$, with multiplication defined by (8) and (9). For example,

$$aba \cdot ab = abab,$$
$$ab \cdot aba = ababa.$$

We shall now give an example of a quasivariety $\mathfrak{L}$ for which the composition homomorphisms in a free composition of $\mathfrak{L}$-systems need not be isomorphisms.

Let $\mathfrak{L}$ be the quasivariety of semigroups satisfying the quasi-identity

$$x^3 = x \to y^3 = y. \tag{10}$$

Let $\mathfrak{A}$ be the infinite cyclic semigroup consisting of the elements $a, a^2, a^3, \ldots$, and let $\mathfrak{B}$ be the cyclic semigroup of order two formed by the elements b and b^2 ($b^3 = b$). Both of these semigroups belong to the class $\mathfrak{L}$ since both elements of $\mathfrak{B}$ satisfy the relation $y^3 = y$, and in $\mathfrak{A}$ we have $x^3 \neq x$ for all x. Let $\mathfrak{C}$ be the $\mathfrak{L}$-free composition $\mathfrak{A} * \mathfrak{B}$ and suppose that $\alpha\colon \mathfrak{A} \to \mathfrak{C}$ and $\beta\colon \mathfrak{B} \to \mathfrak{C}$ are the corresponding composition homomorphisms. From $b^3 = b$, it follows that $(b\beta)^3 = b\beta$, and hence, by (10) the identity $y^3 = y$ holds in $\mathfrak{C}$. In particular, $(a\alpha)^3 = a\alpha$. This shows that α is not an isomorphism since $a^3 \neq a$ in $\mathfrak{A}$.

From the above construction of free compositions in the class of all semigroups, we see that free compositions in this class have the following "normal" properties:

a) *the composition homomorphisms of free compositions of $\mathfrak{K}$-systems are isomorphisms (injectiveness of compositions)*;

b) *if a system $\mathfrak{A}$ is a $\mathfrak{K}$-free product of its subsystems $\mathfrak{A}_i (i \in I)$, then a system $\mathfrak{B} \subseteq \mathfrak{A}$ generated in $\mathfrak{A}$ by arbitrary subsystems $\mathfrak{B}_i$ of the corresponding free factors $\mathfrak{A}_i (i \in I)$ is the $\mathfrak{K}$-free product of the $\mathfrak{B}_i$ (heredity of compositions).*

Besides the class of all semigroups, properties a) and b) are also satisfied by the class of all groups, the class of all abelian groups, the class of all associative rings, the class of all nonassociative rings, the class of Lie rings and many others. However, just as often, one also finds classes which satisfy property a) but not property b). Moreover, among all varieties of groups heretofore considered, only the variety of all groups and the variety of abelian groups satisfy property b). Whether or not there are other varieties of groups with property b) is still an open question.

Let us examine the following popular example. Recall that a group is metabelian if it satisfies the identity

$$x^{-1}y^{-1}xy \cdot z = z \cdot x^{-1}y^{-1}xy. \tag{11}$$

We shall show that the variety of metabelian groups does not have property b).

First we deduce a series of identities which are true in the class of metabelian groups. Suppose $\mathfrak{G}$ is a metabelian group and $u^{-1}v^{-1}uv = w$, where $u, v \in \mathfrak{G}$. According to (11),

$$uw = wu, \quad vw = wv. \tag{12}$$

Raising both sides of $v^{-1}uv = uw$ to an arbitrary power $x = 0, \pm 1, \pm 2, \ldots$ and taking (12) into account, we obtain $v^{-1}u^x v = u^x w^x$ and $u^x v u^{-x} = v w^x$. Raising both sides of the latter equation to an arbitrary power y, we obtain $u^x v^y u^{-x} = v^y w^{xy}$ and hence

$$u^x v^y = v^y u^x w^{xy} \qquad (x, y = 0, \pm 1, \pm 2, \ldots). \tag{13}$$

Now let $\mathfrak{C}$ denote the metabelian free product of cyclic groups $\mathfrak{A}$ and $\mathfrak{B}$ of some prime order p. Suppose a and b are the respective generators of $\mathfrak{A}$ and $\mathfrak{B}$ and $a^{-1}b^{-1}ab = c \in \mathfrak{C}$. According to (13), $a^{-1}b^p a = b^p c^p$, i.e. $c^p = 1$. From (13), it also follows that for any integers x, y, z, x_1, y_1 and z_1 belonging to the set $\{0, 1, \ldots, p-1\}$, the relations

$$\left.\begin{aligned} a^x b^y c^z \cdot a^{x_1} b^{y_1} c^{z_1} &= a^{\overline{x+x_1}} b^{\overline{y+y_1}} c^{\overline{z+z_1-x_1 y}}, \\ (a^x b^y c^z)^{-1} &= a^{-x} b^{-y} c^{-z-xy} = a^{p-x} b^{p-y} c^{\overline{-z-xy}} \end{aligned}\right\} \tag{14}$$

hold, where the bar indicates the residue modulo p. The relations (14) show that the elements of the form

$$g = a^x b^y c^z \qquad (x, y, z = 0, 1, \ldots, p-1) \tag{15}$$

form a subroup in $\mathfrak{C}$. Since this subgroup contains the elements a and b which generate $\mathfrak{C}$, the indicated subgroup coincides with $\mathfrak{C}$; consequently, every element of $\mathfrak{C}$ can be written in the form (15). We wish to show that this notation is unique.

Let $\mathfrak{H}$ denote the collection of triples $\langle x, y, z\rangle$ of numbers $0, 1, \ldots, p-1$. Using formulas (14), we introduce operations of multiplication and inversion in $\mathfrak{H}$ by defining,

$$\left.\begin{aligned} \langle x, y, z\rangle \cdot \langle x_1, y_1, z_1\rangle &= \langle \overline{x + x_1}, \overline{y + y_1}, \overline{z + z_1 - x_1 y}\rangle, \\ \langle x, y, z\rangle^{-1} &= \langle p - x, p - y, \overline{-z - xy}\rangle. \end{aligned}\right\} \tag{16}$$

It is easy to verify by direct computation that the collection $\mathfrak{H}$ is a group under these operations and is also metabelian. The respective triples of the form $\langle x, 0, 0\rangle$ and $\langle 0, x, 0\rangle$ $(x = 0, 1, \ldots, p-1)$ form cyclic subgroups $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in $\mathfrak{H}$. Since the group $\mathfrak{C}$ is by assumption the metabelian free product of its subgroups $\mathfrak{A}$ and $\mathfrak{B}$, the isomorphisms

$$\left.\begin{aligned} \alpha&: \mathfrak{A} \to \mathfrak{H}_1 \qquad (a^x \alpha = \langle x, 0, 0\rangle), \\ \beta&: \mathfrak{B} \to \mathfrak{H}_2 \qquad (b^x \beta = \langle 0, x, 0\rangle) \end{aligned}\right\} \tag{17}$$

must be extendable to a homomorphism $\xi: \mathfrak{C} \to \mathfrak{H}$. From (16) and (17), it follows that

$$c\xi = (a^{-1}b^{-1}ab)\xi = \langle -1, 0, 0\rangle \langle 0, -1, 0\rangle \langle 1, 0, 0\rangle \langle 0, 1, 0\rangle = \langle 0, 0, 1\rangle.$$

and hence

$$(a^x b^y c^z)\xi = \langle x, y, z\rangle \qquad (x, y, z = 0, 1, \ldots, p-1).$$

Distinct triples $\langle x, y, z\rangle$ $(x, y, z = 0, 1, \ldots, p-1)$ are by definition distinct elements of $\mathfrak{H}$. Therefore, products of the form (15) which have distinct sets of exponents x, y, z have distinct values in $\mathfrak{C}$.

Now take the metabelian free product $\mathfrak{D}$ of the same cyclic group $\mathfrak{A}$ of cardinality p and of the cyclic group $\mathfrak{B}_1$ of cardinality p^2. Let d denote the generator of $\mathfrak{B}_1$. The element $b^* = d^p$ generates in $\mathfrak{B}_1$ a cyclic subgroup

$$\mathfrak{B}^* = \{d^p, d^{2p}, \ldots, d^{(p-1)p}, 1\}$$

of order p isomorphic to the group $\mathfrak{B}$ considered above. At the same time, the subgroup $\mathfrak{C}^*$ generated in $\mathfrak{D}$ by the elements of the supgroups $\mathfrak{A}$ and $\mathfrak{B}^*$ is not isomorphic to the group $\mathfrak{C}$. Hence, $\mathfrak{C}^*$ is not the metabelian free product of $\mathfrak{A}$ and $\mathfrak{B}^*$.

Indeed, since $a^{-1}b^{-1}ab = c \neq 1$ in $\mathfrak{C}$, $\mathfrak{C}$ is not commutative. On the other hand, setting $a^{-1}d^{-1}ad = h$ in the group $\mathfrak{D}$, we have $d^{-1}a^p d = a^p h^p$, whence $h^p = 1$. From the formula $a^{-1}d^p a = d^p h^{-p}$, we obtain $a^{-1}d^p a = d^p$, i.e. $d^p a = a d^p$ and $a^x d^{py} = d^{py} a^x$ $(x, y = 0, 1, \ldots, p-1)$.

Thus, the subgroup $\mathfrak{C}^*$ is commutative and hence is not isomorphic to the noncommutative group $\mathfrak{C}$.

12.2. Independent Elements and Free Systems. Let $\mathfrak{A}$ be an algebraic system and let $\mathfrak{K}$ be a class of systems of some fixed signature Ω. A nonempty set S of elements in $\mathfrak{A}$ in said to be *independent in* $\mathfrak{A}$ *relative to the class* $\mathfrak{K}$ ($\mathfrak{K}$-independent) if any mapping of S into any $\mathfrak{K}$-system $\mathfrak{M}$ can be extended to a homomorphism of $\bar{S}$ into $\mathfrak{M}$, where $\bar{S}$ is the subsystem generated by the elements S in $\mathfrak{A}$.

A set S is *independent* in $\mathfrak{A}$ if it is independent relative to the class $\mathfrak{K}$ consisting of the single system $\mathfrak{A}$ (cf. Marczewski [50]).

In these definitions, S may consist of one element. Therefore, it makes sense to speak of independence (or self-independence) of some particular element of the system $\mathfrak{A}$.

Since the definition of $\mathfrak{K}$-independence only involves the subsystem $\bar{S}$ rather than $\mathfrak{A}$ itself, the $\mathfrak{K}$-independence of a set S in $\mathfrak{A}$ is equivalent to the $\mathfrak{K}$-independence of S in any subsystem of $\mathfrak{A}$ containing S. It is also clear that if S is $\mathfrak{K}$-independent, then any subset of S is independent.

The terminology "$\mathfrak{K}$-independent elements" is justified by the following property of such elements.

Theorem 1. *If mutually distinct elements $a_1, \ldots, a_n$ of a system $\mathfrak{A}$ are $\mathfrak{K}$-independent and satisfy in $\mathfrak{A}$ some quasi-atomic relation*

$$P\big(f_1(a_1, \ldots, a_n), \ldots, f_s(a_1, \ldots, a_n)\big) = \mathsf{T}, \tag{1}$$

where $P \in \{\Omega, =\}$ and each $f_i(x_1, \ldots, x_n)$ is a term of signature Ω not necessarily containing all the variables $x_1, \ldots, x_n$, then the identity

$$(\forall x_1 \ldots x_n)\, P\big(f_1(x_1, \ldots, x_n), \ldots, f_s(x_1, \ldots, x_n)\big) \tag{2}$$

is true in $\mathfrak{K}$.

Conversely, suppose S is a certain set of elements of $\mathfrak{A}$ such that if a quasi-atomic relation of the form (1) holds for mutually distinct elements $a_1, \ldots, a_n \in S$, the identity (2) is true in $\mathfrak{K}$. Then S is $\mathfrak{K}$-independent.

Suppose $a_1, \ldots, a_n$ are distinct $\mathfrak{K}$-independent elements, $S_1 = \{a_1, \ldots, a_n\}$ and $c_1, \ldots, c_n$ are certain elements of an arbitrary $\mathfrak{K}$-system $\mathfrak{C}$ (not necessarily distinct). By hypothesis, the mapping

$$\alpha\colon a_i \to c_i \qquad (i = 1, \ldots, n)$$

can be extended to a homomorphism $\gamma\colon \bar{S}_1 \to \mathfrak{C}$. But then, the relation (1) implies

$$\begin{aligned} \mathsf{T} &= P\big(f_1(a_1\gamma, \ldots, a_n\gamma), \ldots, f_s(a_1\gamma, \ldots, a_n\gamma)\big) \\ &= P\big(f_1(c_1, \ldots, c_n), \ldots, f_s(c_1, \ldots, c_n)\big). \end{aligned}$$

Conversely, suppose that a relation of the form (1) involving arbitrary distinct elements $a_1, \ldots, a_n$ of a set S of elements of $\mathfrak{A}$ implies the identity (2) in the class $\mathfrak{K}$. Let α be any mapping of S into an arbitrary $\mathfrak{K}$-system $\mathfrak{C}$. We must extend α to a homomorphism of $\bar{S}$ into $\mathfrak{C}$. Every element $a \in \bar{S}$ can be represented in the form

$$a = f(a_1, \ldots, a_n) \qquad (a_i \in S,\ a_i \neq a_j;\ i \neq j,\ i, j = 1, \ldots, n),$$

where $f(x_1, \ldots, x_n)$ is a term of signature Ω. We introduce the mapping

$$\varphi\colon f(a_1, \ldots, a_n) \to f(a_1\alpha, \ldots, a_n\alpha).$$

We must verify that the mapping φ is well-defined and that it is a homomorphism of $\bar{S}$ into $\mathfrak{C}$, i.e. we must show that if a relation of the form (1) holds, then the relation

$$P\big(f_1(a_1\alpha, \ldots, a_n\alpha), \ldots, f_s(a_1\alpha, \ldots, a_n\alpha)\big) \tag{3}$$

is true.

But this is obvious since the relation (3) is obtained from the identity (2) by the mapping $x_i \to a_i \alpha_i$ $(i = 1, \ldots, n)$.

Corollary 2. *If every finite subset of an infinite set S of elements of a system $\mathfrak{A}$ is $\mathfrak{K}$-independent, then S is also $\mathfrak{K}$-independent.*

Indeed, every quasi-atomic relation of the form (1) involving elements of the set S in fact involves only a finite subset of elements of S which is $\mathfrak{K}$-independent by hypothesis. Hence the identity (2) is true in the class $\mathfrak{K}$.

From the definition of $\mathfrak{K}$-independence, it is immediate that a $\mathfrak{K}$-independent set of elements is also independent relative to any subclass $\mathfrak{L} \subseteq \mathfrak{K}$. However, an independent set of elements relative to the subclass $\mathfrak{L}$ may be dependent relative to the wider class $\mathfrak{K}$. To see this, one merely has to notice, for example that all elements of any system are independent relative to a class consisting only of a trivial system. Nevertheless, the next theorem indicates an important case where independence relative to one class implies independence relative to a wider class.

Recall that for a completely arbitrary class $\mathfrak{L}$, the symbols $S\mathfrak{L}$ and $\prod \mathfrak{L}$ denote, respectively, the classes of all subsystems of $\mathfrak{L}$-systems, and (isomorphic) copies of direct products of $\mathfrak{L}$-systems. In the same way, $H\,\mathfrak{L}$ are all the homomorphic images of $\mathfrak{L}$-systems.

Theorem 3. *If a set S of elements of a system $\mathfrak{A}$ is independent relative to some class $\mathfrak{K}$, then S is also independent relative to the class $HS\prod \mathfrak{K}$.*

To prove this, we merely have to refer to Theorem 1. Indeed, suppose that elements $a_1, \ldots, a_n$ of S satisfy the relation (1). Then, by hypothesis, the identity (2) is true in $\mathfrak{K}$. If the identity is true in $\mathfrak{K}$, then it is true in any direct product of $\mathfrak{K}$-systems, i.e. it is true in $\prod \mathfrak{K}$. If the identity is true in a $\prod \mathfrak{K}$-system, then it is also true in any subsystem of this system, i.e. it is true in $S\prod \mathfrak{K}$. Finally, since the identity is true in any $S\prod \mathfrak{K}$-system, it is also true in any homomorphic image of such a system, i.e. the above identity is true in $HS\prod \mathfrak{K}$.

By the converse in Theorem 1, this shows that the set S is independent relative to the class $HS\prod \mathfrak{K}$.

An algebraic system $\mathfrak{A}$ is said to be *free relative to the class* $\mathfrak{K}$ if $\mathfrak{A}$ is generated by a set of elements S which is independent relative to $\mathfrak{K}$. A set S with this property is called a *$\mathfrak{K}$-free basis* for $\mathfrak{A}$.

A system $\mathfrak{A}$ is said to be *free of rank* $\mathfrak{m}$ *in the class* $\mathfrak{K}$ (symbolically, $\mathfrak{A} = \mathfrak{F}_{\mathfrak{m}}(\mathfrak{K})$) if $\mathfrak{A} \in \mathfrak{K}$ and it has a $\mathfrak{K}$-free basis of cardinality $\mathfrak{m}$. Finally, $\mathfrak{A}$ is said to be *free in itself of rank* $\mathfrak{m}$ if $\mathfrak{A}$ is a free system of rank $\mathfrak{m}$ in the class $\{\mathfrak{A}\}$ consisting only of the system $\mathfrak{A}$.

From the above properties of $\mathfrak{K}$-independent sets of elements, we immediately obtain the following properties of free systems:

a) *Every $\mathfrak{K}$-independent set S of elements of the system $\mathfrak{A}$ generates a subsystem $\bar{S}$ which is free relative to the class $\mathfrak{K}$. S is a $\mathfrak{K}$-free basis of the system $\bar{S}$.*

b) *An algebraic system $\mathfrak{A}$ which is free relative to a class $\mathfrak{K}$ is free relative to any subclass $\mathfrak{L} \subseteq \mathfrak{K}$ and is also free relative to the superclass $HS\prod \mathfrak{K}$.*

Equally clear is the following statement.

Theorem 4. *If a class $\mathfrak{K}$ has free systems of a given rank $\mathfrak{m}$, then they are all isomorphic and any $\mathfrak{K}$-system with a generating set of cardinality $\mathfrak{m}$ is a homomorphic image of a free system of rank $\mathfrak{m}$ in $\mathfrak{K}$.*

In particular, if $\mathfrak{K}$ has free systems of arbitrary rank, then every $\mathfrak{K}$-system $\mathfrak{A}$ is a homomorphic image of a system free in $\mathfrak{K}$ of rank $|\mathfrak{A}|$.

Indeed, suppose $\mathfrak{A}$ and $\mathfrak{B}$ are free systems of rank $\mathfrak{m}$ in $\mathfrak{K}$ and S and T are their free bases of cardinality $\mathfrak{m}$. Since the cardinalities of S and T are equal, there exists a one-to-one mapping $\alpha: S \to T$. By hypothesis, this mapping extends to a homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$. Since $\mathfrak{A}^{\varphi}$ includes the set T which generates $\mathfrak{B}$, φ is an epimorphism. Similarly, the inverse mapping $\alpha^{-1}: T \to S$ extends to an epimorphism $\psi: \mathfrak{B} \to \mathfrak{A}$. The homomorphism $\varphi\psi: \mathfrak{A} \to \mathfrak{A}$ leaves fixed the elements of the set S which generates $\mathfrak{A}$. Hence, $\varphi\psi$ is the identity mapping and so $\psi = \varphi^{-1}$.

Now suppose that $\mathfrak{C}$ is an arbitrary $\mathfrak{K}$-system generated by a set of elements U, where $|U| \leqq |S|$. Take an arbitrary mapping α of S onto U. By hypothesis, α can be extended to a homomorphism $\varphi: \mathfrak{A} \to \mathfrak{C}$. Since $\mathfrak{A}^{\varphi}$ includes the set U generating $\mathfrak{C}$, φ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{C}$.

It was remarked above that in every class, the free systems are determined up to isomorphism by their rank. Do there exist isomorphic free systems of distinct ranks? Before answering this question, we make a few preliminary remarks.

Classes of systems consisting of just one-element systems will be called *trivial*. Trivial classes only contain free systems of rank 1. Also, a system $\mathfrak{A}$ will be free if every formula of the form

$$P(a, \ldots, a) \qquad (P \in \Omega_p,\ a \in \mathfrak{A})$$

is true in $\mathfrak{A}$ if and only if it holds in each system of the class.

A generating set of elements for a system $\mathfrak{A}$ is *minimal* if it has no proper subset which generates $\mathfrak{A}$. From the definition of free basis, we immediately obtain a minimality criterion.

Corollary 5. *Every free basis of a system $\mathfrak{A} = \mathfrak{F}_{\mathfrak{m}}(\mathfrak{K})$ in some class $\mathfrak{K}$ is a minimal generating set in $\mathfrak{A}$.*

Suppose, to the contrary, that some proper part S_1 of the basis S generates $\mathfrak{A}$. Then for every element $a \in S - S_1$, there is a representation of the form

$$a = f(a_1, \ldots, a_n), \tag{4}$$

where $f(x_1, \ldots, x_n)$ is a term of signature Ω and $a_1, \ldots, a_n$ are distinct elements of S_1. According to Theorem 1, (4) implies that the identity

$$x = f(x_1, \ldots, x_n)$$

is true in $\mathfrak{K}$. Since $\mathfrak{A} \in \mathfrak{K}$, this identity is true in $\mathfrak{A}$. Setting $x_1 = a_1, \ldots, x_n = a_n$ and $x = a, a_1$, we obtain $a = a_1$, which is impossible.

Note that if a system $\mathfrak{A}$ has an infinite minimal generating set, then every minimal generating set for $\mathfrak{A}$ will have the same cardinality.*) On the other hand, if $\mathfrak{A}$ has a minimal generating set of cardinality $\mathfrak{m}$, then every system isomorphic to it will also have a minimal generating set of cardinality $\mathfrak{m}$. Comparing these assertions with Corollary 5, we arrive at the following result.

Theorem 6. (Fujiwara [16]). *If in some class $\mathfrak{K}$ the free systems $\mathfrak{F}_\mathfrak{m}$ and $\mathfrak{F}_\mathfrak{n}$ have distinct ranks $\mathfrak{m}$ and $\mathfrak{n}$ and are isomorphic, then $\mathfrak{m}$ and $\mathfrak{n}$ are finite.*

Thus, the matter is reduced to the problem of isomorphism of free systems of finite rank. This is solved by the next theorem.

Theorem 7. (Swierczkowski [68]). *Let $\mathfrak{K}$ be a class of signature Ω containing free systems $\mathfrak{F}_m$ and $\mathfrak{F}_n$ of finite distinct ranks m and n. $\mathfrak{F}_m$ and $\mathfrak{F}_n$ are isomorphic if and only if a collection of identities of the form*

$$f_i\big(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)\big) = x_i \quad (i = 1, \ldots, n), \tag{5}$$

$$g_j\big(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\big) = x_j \quad (j = 1, \ldots, m), \tag{6}$$

is satisfied in $\mathfrak{K}$, where $f_i(x_1, \ldots, x_m)$ and $g_j(x_1, \ldots, x_n)$ are suitable terms in the signature Ω.

Necessity. Suppose $\mathfrak{F}_m \cong \mathfrak{F}_n$. Then $\mathfrak{F}_m$ contains free bases $V = \{v_1, \ldots, v_m\}$ and $W = \{w_1, \ldots, w_n\}$ consisting of m and n elements,

*) Indeed, suppose that the system $\mathfrak{A}$ has an infinite minimal generating set S and that S_1 is an arbitrary minimal generating set for $\mathfrak{A}$. Each element $a \in S$ is representable in the form $a = f(a_1, \ldots, a_n)$, where f is a term in the basic operations of $\mathfrak{A}$ and $a_1, \ldots, a_n$ are distinct elements of S_1. Fix one such expression for each $a \in S$. The set of those elements of S_1 which occur in these expressions for all the $a \in S$, generates the system $\mathfrak{A}$ and hence coincides with S_1. Thus, $|S_1| \leqq \aleph_0 |S|$. Since the cardinality of S is infinite, $\aleph_0 |S| = |S|$ (Section 2.6). This yields $|S_1| \leqq |S|$. Since the set S_1 can not be finite, we also have $|S| \leqq |S_1|$ and hence $|S_1| = |S|$ [*Ed.*].

respectively. Since V and W generate $\mathfrak{F}_m$, there exist representations of the form

$$w_i = f_i(v_1, \ldots, v_m) \qquad (i = 1, \ldots, n),$$
$$v_j = g_j(w_1, \ldots, w_n) \qquad (j = 1, \ldots, m),$$

where f_i and g_j are suitable terms in the signature Ω. Hence, by substitution, we obtain the relations

$$f_i\big(g_1(w_1, \ldots, w_n), \ldots, g_m(w_1, \ldots, w_n)\big) = w_i \qquad (i = 1, \ldots, n), \tag{7}$$

$$g_j\big(f_1(v_1, \ldots, v_m), \ldots, f_n(v_1, \ldots, v_m)\big) = v_j \qquad (j = 1, \ldots, m). \tag{8}$$

Since V and W are $\mathfrak{R}$-independent sets, by Theorem 1, (7) and (8) imply the identities (5) and (6).

Sufficiency. Suppose $V = \{v_1, \ldots, v_m\}$ is a free basis in $\mathfrak{F}_m$. Put

$$w_i = f_i(v_1, \ldots, v_m) \qquad (i = 1, \ldots, n). \tag{9}$$

By the identities (6),

$$v_j = g_j(w_1, \ldots, w_n) \qquad (j = 1, \ldots, m).$$

Therefore, $W = \{w_1, \ldots, w_n\}$ generates the system $\mathfrak{F}_m$. Suppose that the elements of this set are related by

$$P\big(h_1(w_1, \ldots, w_n), \ldots, h_s(w_1, \ldots, w_n)\big) = \mathsf{T} \qquad (P \in \{\Omega_P, =\}).$$

Substituting in place of the elements $w_1, \ldots, w_n$ their representations from (9), we obtain

$$P\big(h_1\big(f_1(v_1, \ldots, v_m), \ldots, f_n(v_1, \ldots, v_m)\big), \ldots\big) = \mathsf{T}.$$

Since the elements $v_1, \ldots, v_m$ are $\mathfrak{R}$-independent, we have in $\mathfrak{F}_m$ the identity

$$P\big(h_1\big(f_1(y_1, \ldots, y_m), \ldots, f_n(y_1, \ldots, y_m)\big), \ldots\big) = \mathsf{T}.$$

Setting $y_j = g_j(x_1, \ldots, x_n)$ in this $(j = 1, \ldots, m)$ and using identities (5), we obtain the identity

$$P\big(h_1(x_1, \ldots, x_n), \ldots, h_s(x_1, \ldots, x_n)\big) = \mathsf{T},$$

showing that the set W is $\mathfrak{R}$-independent.

Thus, the system $\mathfrak{F}_m$ has free bases of cardinalities m and n and hence $\mathfrak{F}_m \cong \mathfrak{F}_n$.

The identities (5) and (6) show that for every system $\mathfrak{A}$ in which they hold, the mapping

$$\langle x_1, \ldots, x_m \rangle \to \langle f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m) \rangle$$

of the set of all m-tuples of elements of $\mathfrak{A}$ onto the set of all n-tuples of these elements is one-to-one.

However, if the cardinality of $\mathfrak{A}$ is finite and equal to d, then the number of m-tuples is equal to d^m, and the number of n-tuples is equal to d^n. For $d \geqq 2$, $m \neq n$ and we have $d^m \neq d^n$. Thus, (5) and (6) can not hold in any finite system with more than one element and therefore, Theorem 7 has the following consequence.

Corollary 8 (Fujiwara [16], Jónsson-Tarski [23]). *If a class $\mathfrak{K}$ has a finite system with more than one element, then all free systems of distinct ranks in $\mathfrak{K}$ are nonisomorphic.*

Above in the proof of Theorem 4, we essentially repeated the argument of Section 11.2 used to establish the uniqueness of a system presented by its defining relations. Comparing the definition of a free system of rank $\mathfrak{m}$ in the class $\mathfrak{K}$ with the definition of a system presented in the class $\mathfrak{K}$ by defining relations, we are immediately convinced of the following fact.

Theorem 9. *If a class $\mathfrak{K}$ has a free system of rank $\mathfrak{m}$, then this system can be presented in $\mathfrak{K}$ by a set of generating symbols of cardinality $\mathfrak{m}$ and the empty set of defining relations. Conversely, if $\mathfrak{K}$ is nontrivial, then a system $\mathfrak{A}$ presented in $\mathfrak{K}$ by a set Z of generating symbols z_i of cardinality $\mathfrak{m}$ and the empty collection of defining relations is a free system of rank $\mathfrak{m}$ in $\mathfrak{K}$.*

We need only verify the second assertion. Consider the canonical mapping $\alpha: Z \to \mathfrak{A}$. From the basic definition of Section 11.2, it follows that the set of elements $z_i\alpha$ generates the system $\mathfrak{A}$ and is $\mathfrak{K}$-independent. We need only show that the cardinality of the set of elements $z_i\alpha$ is equal to $\mathfrak{m}$, i.e. that $z_i\alpha \neq z_j\alpha$ for $i \neq j$. But if it were true that $z_i\alpha = z_j\alpha$ for $i \neq j$, then the independence of the elements $z_i\alpha$ and $z_j\alpha$ in the class $\mathfrak{K}$ would imply the identity $x = y$ and so the class $\mathfrak{K}$ would be trivial.

From Theorem 9 and the appropriate theorems of Sections 11.2 and 11.3, we immediately obtain

Corollary 10. *In every replete nontrivial class $\mathfrak{K}$, there exist free systems of any pre-assigned rank $\mathfrak{m} \geqq 1$. A free system of rank $\mathfrak{m}$ in any subclass $\mathfrak{L}$ of $\mathfrak{K}$ is an $\mathfrak{L}$-replica of the free system of the same rank $\mathfrak{m}$ in $\mathfrak{K}$.*

A free system in the class of all systems of a given signature Ω is called *absolutely free*. From Corollary 10, we find that a free system of rank $\mathfrak{m}$ in an arbitrary class $\mathfrak{K}$ is a $\mathfrak{K}$-replica of an absolutely free system of rank $\mathfrak{m}$.

Section 12.1 showed that if systems $\mathfrak{A}_\mu$ ($\mu \in M$) are presented in a class $\mathfrak{K}$ by pairwise disjoint collections Z_μ of generating symbols and corresponding collections T_μ of defining relations, then the system $\mathfrak{A}$ presented in $\mathfrak{K}$ by $\bigcup Z_\mu$, the union of the generating symbols, and $\bigcup T_\mu$, the union of the defining relations, is the $\mathfrak{K}$-free composition of the systems $\mathfrak{A}_\mu$. Taking T_μ to be the empty collection and applying Theorem 9, we deduce.

Corollary 11. *A free system of rank* $\mathfrak{m}$ *in the class* $\mathfrak{K}$ *is isomorphic to the* $\mathfrak{K}$*-free* $\mathfrak{m}$-th *power of a free system of rank* 1.

From Section 12.1, we obtain in particular that the free semigroup (in the class of all semigroups) with free basis $\{z_i \mid i \in I\}$ is the semigroup of nonempty words written without parentheses in the alphabet $\{z_i \mid i \in I\}$ with concatenation of words as operation. Analogously, the free group (in the class of all groups) with free basis $\{z_i \mid i \in I\}$ consists of the words in the alphabet $\{z_i, z_i^{-1} \mid i \in I\}$ which have no adjacent occurrences of "inverse" symbols z_i and z_i^{-1}. The multiplication here is again concatenation of words, but with subsequent deletion of adjacent mutally inverse symbols (cf. Kuroš [31]).

By Corollary 11, for any cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ we have the isomorphism

$$\mathfrak{F}_\mathfrak{m} * \mathfrak{F}_\mathfrak{n} \cong \mathfrak{F}_{\mathfrak{m}+\mathfrak{n}},$$

and therefore, *the collection of all free systems of finite rank is a cyclic semigroup under free composition providing isomorphic systems are identified.* The generator of this semigroup is the free system of rank 1.

In Section 3.1, it was proved that a cyclic semigroup is either infinite, in which case all powers of its generating element are distinct, or the semigroup is finite and there exist natural numbers k and l ($1 \leqq k < l$) with the property that the positive powers u^m and u^n ($m < n$) of its generating element u are equal if and only if

$$k \leq m, \qquad n - m = r(l - k). \tag{10}$$

Consequently, for any class $\mathfrak{K}$ with free systems, either all the free systems of distinct finite ranks are not isomorphic, or there exist natural numbers k and l ($1 \leqq k < l$), such that free systems $\mathfrak{F}_m$ and $\mathfrak{F}_n$ of finite ranks m and n ($m < n$) are isomorphic if and only if the conditions (10) are fulfilled. By Corollary 8, the first case happens, for instance, in the

class of all groups, the class of all semigroups and in other classes. It remains to prove that for arbitrary pairs of integers k and l ($1 \leqq k < l$), the second case also happens in some class.

Theorem 12 (Swierczkowski [68]).*) *Let $\mathfrak{S}_{k,l}$ ($1 \leqq k < l$) be a variety whose signature consists of k-ary operator smybols F_i and l-ary operator symbols G_j ($i = 1, \ldots, l$; $j = 1, \ldots, k$). Let $\mathfrak{S}_{k,l}$ be defined by the identities*

$$F_i(G_1(x_1, \ldots, x_l), \ldots, G_k(x_1, \ldots, x_l)) = x_i \qquad (i = 1, \ldots, l),$$
$$G_j(F_1(x_1, \ldots, x_k), \ldots, F_l(x_1, \ldots, x_k)) = x_j \qquad (j = 1, \ldots, k).$$

Then the free algebras in $\mathfrak{S}_{k,l}$ of finite rank form a cyclic semigroup under free composition with index pair $\langle k, l \rangle$.

From Theorem 7 it follows that free algebras in the variety $\mathfrak{S}_{k,l}$ of rank k and l are certainly isomorphic. It remains to show that $\mathfrak{F}_l \cong \mathfrak{F}_i$ ($i < l$) implies that $i = k$. This requires a more detailed study of the structure of free algebras in $\mathfrak{S}_{k,l}$. For that reason, the proof of Theorem 12 will be omitted.

12.3 Amalgamated Compositions. Let $\mathfrak{K}$ be a fixed class of algebraic systems of signature Ω, and let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary $\mathfrak{K}$-systems which have a nonempty intersection (Fig. 1). Does there exist a system $\mathfrak{H}$ in $\mathfrak{K}$

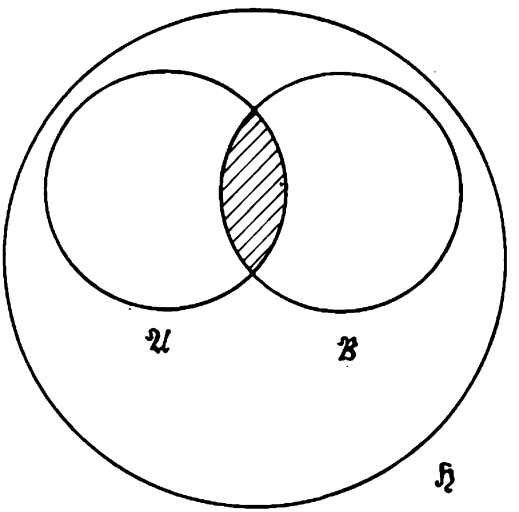

Fig. 1

*) The isomorphism of free algebras of finite rank in the variety $\mathfrak{S}_{1,2}$ was first pointed out by Jónsson and Tarski [23] [*Ed.*].

of which both $\mathfrak{A}$ and $\mathfrak{B}$ are subsystems? If such systems $\mathfrak{H}$ exist, is it possible to pick out a special system from among them (by imposing additional but general conditions) which is in a certain sense most free? The answer to these questions is embodied in the construction given below, known as free composition with amalgamated subsystems. Initially, we only remark that a positive answer to the first of the above questions obviously makes it necessary that the intersection of the given systems $\mathfrak{A}$ and $\mathfrak{B}$ be a subsystem of each of them. This will be assumed below. Moreover, instead of two systems, we shall consider an arbitrary collection of systems. But for brevity, we shall assume that the intersection of any two of the systems is one and the same system.

Thus, suppose we are given a sequence $\mathfrak{A}_i$ $(i \in I)$ of systems of signature Ω and an arbitrary class $\mathfrak{K}$ of systems of this signature. Furthermore, suppose that in each $\mathfrak{A}_i$ there is a distinguished subsystem $\mathfrak{C}_i$. We shall assume that these subsystems are isomorphic and that the isomorphisms between them $\gamma_{ij}\colon \mathfrak{C}_i \to \mathfrak{C}_j$ are fixed and satisfy the compatibility conditions

$$\gamma_{ij}\gamma_{jk} = \gamma_{ik}, \quad \gamma_{ii} = \varepsilon_i \quad (i, j, k \in I) \tag{1}$$

(ε_i is the identity mapping of $\mathfrak{C}_i$ onto itself). The isomorphisms γ_{ij} identify the subsystems $\mathfrak{C}_i (i \in I)$ allowing them to be viewed as the same subsystem $\mathfrak{C}$ of each of the given systems $\mathfrak{A}_i$. For the purpose of generality, we shall not exclude the possibility that $\mathfrak{C}$ or certain of the given systems $\mathfrak{A}_i$ are empty. The identified subsystems $\mathfrak{C}_i$ are usually called amalgamated.

The $\mathfrak{K}$-system $\mathfrak{H}$ is said to be a *$\mathfrak{K}$-composition of the systems $\mathfrak{A}_i$ with the amalgamations* (1) *of the subsystems* $\mathfrak{C}_i$ if there exist homomorphisms

$$\alpha_i\colon \mathfrak{A}_i \to \mathfrak{H} \quad (i \in I)$$

satisfying:

a) the homomorphisms α_i are compatible with the amalgamations, i.e. $\gamma_{ij}\alpha_j = \varepsilon_i\alpha_i$ $(i, j \in I)$;

b) $\mathfrak{H}$ is generated by its subsystems $\mathfrak{A}_i\alpha_i$;

c) for arbitrary homomorphisms $\alpha_i'\colon \mathfrak{A}_i \to \mathfrak{M}$ into any $\mathfrak{H}$-system $\mathfrak{M}$ which are compatible with the amalgamations, there exists a homomorphism $\xi\colon \mathfrak{H} \to \mathfrak{M}$ satisfying $\alpha_i' = \alpha_i\xi (i \in I)$.

If $\mathfrak{C}_i = \varnothing$, then the above definition becomes the definition of $\mathfrak{K}$-composition of systems without amalgamations. The homomorphisms α_i are called the *composition* homomorphisms of the "factors" $\mathfrak{A}_i$ into their amalgamated $\mathfrak{K}$-composition $\mathfrak{H}$.

A system $\mathfrak{A}$ is said to be the *$\mathfrak{K}$-product* (or *$\mathfrak{K}$-free product*) of its subsystems $\mathfrak{A}_i$ over the (amalgamated) subsystem $\mathfrak{C} \subseteq \bigcap \mathfrak{A}_i$ if $\mathfrak{A}$ is the

$\mathfrak{K}$-free composition of the $\mathfrak{A}_i$ with the identity mappings of $\mathfrak{C}$ onto $\mathfrak{C}$ as amalgamations and the inclusion mappings of $\mathfrak{A}_i$ into $\mathfrak{A}$ as composition homomorphisms.

From these definitions, we immediately obtain

Corollary 1. *Every system $\mathfrak{H}$ which is the $\mathfrak{K}$-composition of systems $\mathfrak{A}_i$ $(i \in I)$ with the amalgamations* (1) *and composition homomorphisms $\alpha_i\colon \mathfrak{A}_i \to \mathfrak{H}$ is at the same time a $\mathfrak{K}$-product of its subsystems $\mathfrak{A}_i\alpha_i$ over the amalgamated subsystem $\mathfrak{C}_i\alpha_i$.*

If the system $\mathfrak{A}$ is the $\mathfrak{K}$-free product of its subsystems $\mathfrak{A}_i$ $(i \in I)$ over the amalgamated subsystem $\mathfrak{C} \subseteq \bigcap \mathfrak{A}_i$, then $\mathfrak{A}$ is the $\mathfrak{K}$-free product of the $\mathfrak{A}_i$ over any larger system $\mathfrak{D} \supseteq \mathfrak{C}$ which is contained in the intersection of all $\mathfrak{A}_i$.

The existence of $\mathfrak{K}$-compositions with amalgamations and their uniqueness follow from the standard presentation of these compositions with the help of defining relations constructed in the following way. Consider a sequence of systems $\mathfrak{A}_i$ $(i \in I)$ with amalgamations (1). With each element $a \in \mathfrak{A}_i$, associate a special symbol $z_a{}^i$ and denote by $D_a(\mathfrak{A}_i)$ the atomic diagram of $\mathfrak{A}_i$ expressed in terms of the $z_a{}^i$. Let $\mathcal{A}_\gamma$ denote the collection of equations

$$z_a{}^i = z^i_{a\gamma ij} \qquad (i, j \in I)$$

characterizing the amalgamation of the subsystems $\mathfrak{A}_i$. Comparing the definition of amalgamated composition with the presentation of the system by means of the given defining relations, we conclude the following fact.

A system $\mathfrak{H}$ defined in a class $\mathfrak{K}$ by the set of generating symbols $z_a{}^i$ $(i \in I, a \in \mathfrak{A}_i)$ and the set of relations $\mathcal{A}_\gamma \cup D_a(\mathfrak{A}_i)$ coincides with the $\mathfrak{K}$-composition of the systems $\mathfrak{A}_i$ with amalgamations (1).

This implies that every replete class $\mathfrak{K}$ has amalgamated $\mathfrak{K}$-compositions of arbitrary systems and that if in some $\mathfrak{K}$, a $\mathfrak{K}$-composition of systems with given amalgamations exists, then it is uniquely defined up to isomorphisms which preserve the composition homomorphisms. Furthermore, as in Section 12.1, it is easy to deduce that amalgamated compositions are associative under an appropriate interpretation of associativity. For $\mathfrak{K}$-products, this means that if a system $\mathfrak{A}$ is the $\mathfrak{K}$-product of its subsystems $\mathfrak{A}_i (i \in I)$ over the amalgamated subsystem $\mathfrak{C} \subseteq \bigcap \mathfrak{A}_i$ and $I = \bigcup I_\mu$ $(\mu \in M)$ is a partition of the set I, then $\mathfrak{A}$ is the $\mathfrak{K}$-product of subsystems $\mathfrak{B}_\mu$ over $\mathfrak{C}$, where $\mathfrak{B}_\mu$ is the subsystem generated in $\mathfrak{A}$ by the subsystems $\mathfrak{A}_i (i \in I_\mu)$, each subsystem $\mathfrak{B}_\mu$ itself being a $\mathfrak{K}$-product over $\mathfrak{C}$ of its subsystems $\mathfrak{A}_i (i \in I_\mu)$.

In Section 12.1, the $\mathfrak{K}$-composition of the systems $\mathfrak{A}_i$ was called injective if all the canonical homomorphisms belonging to this compo-

sition were isomorphisms, and hence the $\mathfrak{K}$-composition of the $\mathfrak{A}_i$ could be identified with their $\mathfrak{K}$-product. Analogously, we shall say that the $\mathfrak{K}$-free composition of systems $\mathfrak{A}_i (i \in I)$ with nonempty amalgamations $\gamma_{ij}: \mathfrak{C}_i \to \mathfrak{C}_j$ is *injective* if all the composition homomorphisms α_i are isomorphisms and $\mathfrak{A}_i^{a_i} \cap \mathfrak{A}_j^{a_j} = \mathfrak{C}_i^{a_i}$ $(i \in I)$.

A subsystem $\mathfrak{C}$ of $\mathfrak{A}$ is called a *retract* of $\mathfrak{A}$ if there is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{C}$ which is the identity on all the elements of $\mathfrak{C}$.

Theorem 2. *A $\mathfrak{K}$-free composition $\mathfrak{H}$ of the $\mathfrak{K}$-systems $\mathfrak{A}$ and $\mathfrak{B}$ with the amalgamation $\sigma: \mathfrak{C} \to \mathfrak{D}$ of their $\mathfrak{K}$-subsystems is injective if $\mathfrak{C}$ is a retract of $\mathfrak{A}$ and $\mathfrak{D}$ is a retract of $\mathfrak{B}$.*

Suppose that $\alpha: \mathfrak{A} \to \mathfrak{H}$ and $\beta: \mathfrak{B} \to \mathfrak{H}$ are the corresponding composition homomorphisms. Consider the identity homomorphism $\varepsilon: \mathfrak{B} \to \mathfrak{B}$ and the retraction homomorphism $\varrho: \mathfrak{A} \to \mathfrak{C}$. According to the definition, for the homomorphisms $\varrho\sigma: \mathfrak{A} \to \mathfrak{B}$ and $\varepsilon: \mathfrak{B} \to \mathfrak{B}$, there must exist a homomorphism $\xi: \mathfrak{H} \to \mathfrak{B}$ satisfying $\varrho\sigma = \alpha\xi$ and $\varepsilon = \beta\xi$. Since $\mathfrak{C}^a \subseteq \mathfrak{A}^a \cap \mathfrak{B}^\beta$, we have

$$\mathfrak{C}^{a\xi} \subseteq (\mathfrak{A}^a \cap \mathfrak{B}^\beta)^\xi \subseteq \mathfrak{A}^{a\xi} \cap \mathfrak{B}^{\beta\xi} = \mathfrak{D} = \mathfrak{C}^{a\xi}.$$

This shows that the subsets $\mathfrak{C}^a$ and $\mathfrak{A}^a \cap \mathfrak{B}^\beta$ of $\mathfrak{B}^\beta$ have the same image under the mapping $\xi: \mathfrak{B}^\beta \to \mathfrak{B}$. Since the latter mapping is one-to-one, we have $\mathfrak{C}^a = \mathfrak{A}^a \cap \mathfrak{B}^\beta$.

Let us now consider the so-called *absolute amalgamations*, i.e. amalgamated compositions in the class of all systems of a given signature. Here we have the basic

Theorem 3. *A system $\mathfrak{A}$ of signature Ω is an absolutely free product of its subsystems $\mathfrak{A}_i$ $(i \in I)$ over the amalgamated subsystem $\mathfrak{C} \subseteq \bigcap \mathfrak{A}_i$ if and only if all the following conditions are satisfied:*

a) *the intersection of any two subsystems $\mathfrak{A}_i$ and $\mathfrak{A}_j$ $(i \neq j)$ is $\mathfrak{C}$;*

b) *$\mathfrak{A}$ is generated by the elements of the subsystems $\mathfrak{A}_i$;*

c) *if a relation of the form*

$$F_\lambda(w_1, \ldots, w_n) = w \in \mathfrak{A}_i \qquad (F_\lambda \in \Omega_F; w_1, \ldots, w_n \in \mathfrak{A})$$

holds in $\mathfrak{A}$, then the set $\{w_1, \ldots, w_n\}$ lies entirely in some $\mathfrak{A}_j$;

d) *if an equation of the form*

$$F_\lambda(u_1, \ldots, u_n) = F_\mu(v_1, \ldots, v_m)$$
$$(F_\lambda, F_\mu \in \Omega_F; u_1, \ldots, u_n, v_1, \ldots, v_m \in \mathfrak{A})$$

holds in $\mathfrak{A}$, then either $\{u_1, \ldots, u_n\} \subseteq \mathfrak{A}_i$, $\{v_1, \ldots, v_m\} \subseteq \mathfrak{A}_j$ for suitable i, j, or $\lambda = \mu$, $n = m$, and $u_1 = v_1, \ldots, u_n = v_n$;

e) *if a relation of the form*

$$P(u_1, \ldots, u_n) = \mathsf{T} \qquad (P \in \Omega_P;\ u_1, \ldots, u_n \in \mathfrak{A})$$

holds in $\mathfrak{A}$, *then* $\{u_1, \ldots, u_n\} \subseteq \mathfrak{A}_i$ *for some* $i \in I$.

Sufficiency. Suppose we are given homomorphisms $\alpha_i' : \mathfrak{A}_i \to \mathfrak{M}$ $(i \in I)$ into an arbitrary system $\mathfrak{M}$ of signature Ω which are compatible on the amalgamated subsystem $\mathfrak{C}$, i.e. which satisfy

$$c\alpha_i' = c\alpha_j' \qquad (c \in \mathfrak{C};\ i, j \in I).$$

We introduce a binary relation $\xi \subseteq \mathfrak{A} \times \mathfrak{M}$, putting $\langle u, v \rangle \in \xi$ if for some (possibly improper) term $f(x_1, \ldots, x_n)$ of signature Ω, we have for suitable $w_1, \ldots, w_n \in \bigcup \mathfrak{A}_i$

$$u = f(w_1, \ldots, w_n), \qquad v = f(w_1\varphi, \ldots, w_n\varphi), \tag{2}$$

where φ denotes the mapping defined by the conditions $w\varphi = w\alpha_i'$ $(w \in \mathfrak{A}_i)$.

From condition b), it follows that the left-hand part of the domain of the relation ξ coincides with $\mathfrak{A}$. From conditions c) and d) and (2), it follows that

$$u \xi v \ \& \ u \xi v' \Rightarrow v = v' \quad (u \in \mathfrak{A};\ v, v' \in \mathfrak{M}),$$

i.e. ξ is a mapping of $\mathfrak{A}$ into $\mathfrak{M}$. Finally, from the condition e) and (2), we see that ξ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{M}$ coinciding on $\mathfrak{A}_i$ with the homomorphism α_i'.

Necessity. To prove necessity, we must construct a system $\mathfrak{A}'$ of signature Ω containing subsystems $\mathfrak{A}_i'$ which satisfy conditions a)—e) and which admit isomorphisms $\psi_i : \mathfrak{A}_i \to \mathfrak{A}_i'$ compatible on the intersections. Indeed, since $\mathfrak{A}$ is an absolutely free amalgamated product, the isomorphisms ψ_i must have a common extension to a homomorphism $\psi : \mathfrak{A} \to \mathfrak{A}'$. By the above sufficiency proof of conditions a)—e), the system $\mathfrak{A}'$ is simultaneously with $\mathfrak{A}$ the absolutely free amalgamated product of the given subsystems and hence the homomorphism ψ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$ taking the subsystems $\mathfrak{A}_i$ to the subsystems $\mathfrak{A}_i'$. Since the properties a)—e) are preserved under isomorphism, the collection of subsystems $\mathfrak{A}_i$ of $\mathfrak{A}$ must satisfy them.

Let us now turn to the construction of the required system $\mathfrak{A}'$. Here, in order to prove simultaneously that an arbitrary absolutely free amalgamated composition is injective, we shall generalize the problem slightly by assuming that we have not the subsystems $\mathfrak{A}_i$ of an already existing system $\mathfrak{A}$, but simply a sequence of systems $\mathfrak{A}_i$ $(i \in I)$ with amalgamations of the form (1). We take a fixed $k \in I$ and

set $\mathfrak{C}_k = \mathfrak{C}$. Using the process of Section 2.2 for embedding one system into another, we may construct to each system $\mathfrak{A}_i$ a system $\mathfrak{A}_i^*$ isomorphic to it which contains the subsystem $\mathfrak{C}$ and admits the isomorphism $\varphi_i: \mathfrak{A}_i \to \mathfrak{A}_i^*$ taking $\mathfrak{C}_i$ into $\mathfrak{C}$ and such that the original amalgamations $\gamma_{ij}: \mathfrak{C}_i \to \mathfrak{C}_j$ become the identity mappings of $\mathfrak{C} \subseteq \mathfrak{A}_i^*$ onto $\mathfrak{C} \subseteq \mathfrak{A}_j^*$. Moreover, we may assume that the systems $\mathfrak{A}_i^*$ and $\mathfrak{A}_j^*$ $(i \neq j)$ have no elements in common except those of the subsystem $\mathfrak{C}$. In other words, this means that instead of an arbitrarily given collection of systems $\mathfrak{A}_i$ with amalgamations γ_{ij}, we can consider a collection of systems $\mathfrak{A}_i^*$ any two of which intersect only in one and the same system $\mathfrak{C}$ with the amalgamations the identity mappings of $\mathfrak{C}$ onto $\mathfrak{C}$. Given the collection $\mathfrak{A}_i^*$ $(i \in I)$, we wish to construct a system $\mathfrak{A}^*$ which will later turn out to be the desired absolutely free amalgamated composition of the $\mathfrak{A}_i^*$.

With each element $a \in \bigcup \mathfrak{A}_i^*$, we associate a special symbol z_a. The atomic term

$$F_\lambda(z_{a_1}, \ldots, z_{a_n}) \qquad (F_\lambda \in \Omega_F;\ a_1, \ldots, a_n \in \bigcup \mathfrak{A}_i^*)$$

will be called normal if the set $\{a_1, \ldots, a_n\}$ does not lie entirely in any one of the $\mathfrak{A}_i^*$. An arbitrary term $f(z_{a_1}, \ldots, z_{a_n})$ of signature Ω will be called normal if all its atomic subterms are normal. Denote by A^* the collection of all normal terms and all symbols z_a, $a \in \bigcup \mathfrak{A}_i^*$.

We define operations F^* and predicates P^* $(F, P \in \Omega)$ on A^* letting

$$F^*(u_1, \ldots, u_m) = F(u_1, \ldots, u_m) \qquad (u_1, \ldots, u_m \in A^*)$$

if the term $F(u_1, \ldots, u_m)$ is normal. If this term is not normal, then it has the form $F(z_{a_1}, \ldots, z_{a_m})$, where $\{a_1, \ldots, a_m\} \subseteq \mathfrak{A}_i$. Then by definition, we put

$$F^*(z_{a_1}, \ldots, z_{a_m}) = z_a \qquad \big(a = F(a_1, \ldots, a_m)\big).$$

Finally, we put

$$P^*(z_{a_1}, \ldots, z_{a_m}) = P(a_1, \ldots, a_m)$$

if $\{a_1, \ldots, a_m\}$ is contained in some $\mathfrak{A}_i$, and we put $P^*(u_1, \ldots, u_m) = \mathsf{F}$ for all other values of $u_1, \ldots, u_m$ in A^*.

The system $\mathfrak{A}^* = \langle A^*, \Omega \rangle$ defined in this way contains the subsystems $\mathfrak{A}_i'$ generated by the symbols z_a $(a \in \mathfrak{A}_i^*)$. It is clear that these subsystems satisfy the requirements a)—e) in $\mathfrak{A}^*$ and that the mappings $a \to z_a$ are isomorphisms of the $\mathfrak{A}_i^*$ onto $\mathfrak{A}_i'$ compatible on intersections, as required.

Thus, we have proved not only Theorem 3 but also the injectiveness of absolutely free compositions with amalgamations.

Putting $\mathfrak{C} = \varnothing$ in Theorem 3, we obtain necessary and sufficient conditions for a system $\mathfrak{A}$ to be an absolutely free product of its subsystems $\mathfrak{A}_i$.

Theorem 4. *Suppose the system $\mathfrak{A}$ is the absolutely free product of its subsystems $\mathfrak{A}_i (i \in I)$ over the amalgamated subsystem $\mathfrak{C} \subseteq \bigcap \mathfrak{A}_i$; suppose $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$ which contains $\mathfrak{C}$. Denote by $\mathfrak{M}$ the subsystem generated in $\mathfrak{A}$ by the subsystems $\mathfrak{B}_i = \mathfrak{B} \cap \mathfrak{A}_i$. Then $\mathfrak{M}$ is the absolutely free product of its subsystems $\mathfrak{B}_i$ over the amalgamated subsystem $\mathfrak{C}$ and $\mathfrak{B}$ either coincides with $\mathfrak{M}$ or is an absolutely free product of $\mathfrak{M}$ and a suitable absolutely free subsystem $\mathfrak{B}$ of $\mathfrak{A}$.*

First, let us prove that $\mathfrak{M} \cap \mathfrak{A}_i = \mathfrak{B}_i$. Recall that an atomic term $f(w_1, \ldots, w_n)$ is normal if the set $\{w_1, \ldots, w_n\}$ does not lie entirely in one system $\mathfrak{A}_i$. If the above term is not normal, then its value is an element of that system $\mathfrak{A}_i$ which contains all the elements $\{w_1, \ldots, w_n\}$. An arbitrary (not necessarily atomic) term $f(w_1, \ldots, w_n)$ $(w_1, \ldots, w_n \in \cup \mathfrak{A}_i)$ will be called normal if all its atomic subterms are normal. By hypothesis, the system $\mathfrak{M}$ is generated by the subsystems $\mathfrak{B}_i$. Hence each of its elements w is the value of some suitable term $f(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n \in \cup \mathfrak{B}_i$. Replacing non-normal atomic subterms of this term by their values and repeating this operation a sufficient number of times, we find that w is the value of a normal term of the above form or $w \in \mathfrak{B}_i$. By properties c) and d), the value of a term containing even one normal subterm can not belong to $\cup \mathfrak{A}_i$. Suppose now that $\mathfrak{M} \cap \mathfrak{A}_i \neq \mathfrak{B}_i$ and $w \notin \mathfrak{B}_i$, $w \in \mathfrak{M} \cap \mathfrak{A}_i$. Then $w = f(w_1, \ldots, w_n) \in \mathfrak{A}_i$, where $w_1, \ldots, w_n \in \cup \mathfrak{B}_i$ and f is a normal term in the $w_1, \ldots, w_n$, which contradicts the statement just made.

In a similar way, it can be verified that the subsystems $\mathfrak{B}_i$ have the properties a)—e) of Theorem 3 in $\mathfrak{M}$. Therefore, $\mathfrak{M}$ is the absolutely free product of the $\mathfrak{B}_i$ over $\mathfrak{C}$.

If $\mathfrak{B} = \mathfrak{M}$, then the theorem is proved. Let us assume that this is not the case. An element $b \in \mathfrak{B}$ will be called decomposable if $b \in \cup \mathfrak{B}_i$, or $b = F_\alpha(b_1, \ldots, b_m)$ for suitable $b_1, \ldots, b_m \in \mathfrak{B}$ and $F_\alpha \in \Omega$. Suppose $\{v_\lambda \mid \lambda \in L\} = V$ is the set of all indecomposable elements of $\mathfrak{B}$. Let $\overline{V}$ be the subsystem generated in $\mathfrak{B}$ by the set V. We shall show that $\overline{V}$ is an absolutely free system with free generators v_λ. By Section 12.2, to this end, it is sufficient to prove that in $\overline{V}$,

1) a relation of the form

$$F_\alpha(w_1, \ldots, w_m) = v_\lambda \qquad (w_1, \ldots, w_m \in \overline{V},\ v_\lambda \in V)$$

is impossible and that

2) the relation

$$F_\alpha(w_1, \ldots, w_m) = F_\beta(w_1', \ldots, w_m') \qquad (w_1, \ldots, w_m, w_1', \ldots, w_m' \in \overline{V})$$

implies $\alpha = \beta$ and $w_1 = w_1', \ldots, w_m = w_m'$.

But the relation (1) implies that the element v_i is decomposable, which contradicts the definition of V, and condition 2) follows from Theorem 3 since $\mathfrak{M} \cap \overline{V} = \varnothing$.

We shall now prove that $\mathfrak{B} = \mathfrak{M} * \overline{V}$. To this end, we must verify that the product $\mathfrak{M} * \overline{V}$ has properties a)—e) of Theorem 3. We show that $\mathfrak{B}$ is generated by the systems $\mathfrak{M}$ and $\overline{V}$. Each element of $\mathfrak{B}$ is representable in the form of a value of a certain term $f(a_1, \ldots, a_m)$ in elements $a_1, \ldots, a_m$ of the set $\bigcup \mathfrak{A}_i$. Suppose that $\mathfrak{B}$ has elements which can not be expressed in terms of elements of $\mathfrak{M}$ and $\overline{V}$. Let b be the element representable as a term with a minimal number of operator symbols occurring in its representation $f(a_1, \ldots, a_m)$. If this term has no operator symbols or if the set $a_1, \ldots, a_m$ belongs entirely to some system $\mathfrak{A}_i$, then $b \in \mathfrak{M}$, which contradicts our assumption. Therefore, b is representable in the form

$$b = F_\alpha(u_1, \ldots, u_m), \tag{3}$$

where the set of $u_1, \ldots, u_m$ does not lie in any one of the $\mathfrak{A}_i$. If the elements $u_1, \ldots, u_m$ belong to the system $\mathfrak{B}$, then they must, by hypothesis, be expressible in terms of the elements of $\mathfrak{M}$ and $\overline{V}$ since the terms $u_1, \ldots, u_m$ have fewer embedments of operator symbols than the term b. But in that case, b is also expressible in terms of the elements of $\mathfrak{M}$ and $\overline{V}$.

Suppose that not all the elements $u_1, \ldots, u_m$ in the representation (3) belong to $\mathfrak{B}$. Then the set $\{u_1, \ldots, u_m\}$ does not lie entirely in any one $\mathfrak{A}_i$. By Theorem 3, this implies that the representation (3) is unique and, in particular, b can not be decomposed in $\mathfrak{B}$. Hence $b \in V$, which again contradicts the assumption.

Thus, $\mathfrak{B}$ is generated by the subsystems $\mathfrak{M}$ and $\overline{V}$ and $\mathfrak{M} \cap \overline{V} = \varnothing$. The remaining properties c)—e) in Theorem 3 are just as easy to check. For example, if

$$w = F_\alpha(w_1, \ldots, w_m) \in \overline{V} \qquad (w_1, \ldots, w_m \in \mathfrak{B}),$$

then w is decomposable in $\mathfrak{B}$, $w \notin V$ and hence

$$F_\alpha(w_1, \ldots, w_m) = F_\beta(w_1', \ldots, w_m') \qquad (w_1', \ldots, w_m' \in \overline{V}).$$

Since $\{w_1', \ldots, w_m'\}$ does not lie in any single factor $\mathfrak{A}_i$, we have $\alpha = \beta$ and $w_1 = w_1', \ldots, w_m = w_m'$.

In an analogous way, we can also prove

Theorem. 5. *Suppose that the system* $\mathfrak{A}$ *is an absolutely free product of its subsystems* $\mathfrak{A}_i (i \in I)$ *over the amalgamated system* $\mathfrak{C}$. *Suppose* $\mathfrak{B}$ *is a subsystem of* $\mathfrak{A}$ *not containing elements of* $\mathfrak{C}$. *Then* $\mathfrak{B}$ *is an absolutely free product of the intersections* $\mathfrak{B} \cap \mathfrak{A}_i$ $(i \in I)$ *and, possibly, one other absolutely free subsystem* $\mathfrak{B}$ *of* $\mathfrak{A}$.

We also note that the additional absolutely free factors $\mathfrak{B}$ mentioned in Theorems 4 and 5 do not contain elements of the basic factors $\mathfrak{A}_i$.

Theorem. 6 *If a system* $\mathfrak{A}$ *is the absolutely free product of the pair of of systems* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *and simultaneously of the pair of systems* $\mathfrak{A}_1$ *and* $\mathfrak{A}_3$ *over the same amalgamated system* $\mathfrak{C}$, *then* $\mathfrak{A}_2 = \mathfrak{A}_3$.

We shall call an atomic term $F(u_1, \ldots, u_m)$ $(F \in \Omega;\ u_1, \ldots, u_m \in \mathfrak{A})$ $\mathfrak{A}_1$-normal if $u_i \in \mathfrak{A}_1 - \mathfrak{C}$ and $u_j \neq \mathfrak{A}_1$ for certain $i, j = 1, \ldots, m$. An arbitrary term $f(u_1, \ldots, u_n)$ $(u_1, \ldots, u_n \in \mathfrak{A})$ will be called $\mathfrak{A}_1$-normal if it contains an $\mathfrak{A}_1$-normal atomic subterm. Applying Theorem 3 to the decompostions $\mathfrak{A} = \mathfrak{A}_1 \underset{\mathfrak{C}}{*} \mathfrak{A}_2 = \mathfrak{A}_1 \underset{\mathfrak{C}}{*} \mathfrak{A}_3$, we see that the values of $\mathfrak{A}_1$-normal terms do not belong to any one of the subsystems $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$. On the other hand, since $\mathfrak{A}$ is generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$, every element of $\mathfrak{A}$ not in $\mathfrak{A}_1 \cup \mathfrak{A}_2$ is the value of a suitable $\mathfrak{A}_1$-normal term. Thus, denoting by M the collection of values of all $\mathfrak{A}_1$-normal terms, we obtain $\mathfrak{A}_2 = (\mathfrak{A} - M) - (\mathfrak{A}_1 - \mathfrak{C})$. Applying the same reasoning to the decomposition $\mathfrak{A} = \mathfrak{A}_1 \underset{\mathfrak{C}}{*} \mathfrak{A}_3$, we obtain $\mathfrak{A}_3 = (\mathfrak{A} - M) - (\mathfrak{A}_1 - \mathfrak{C})$, i.e. $\mathfrak{A}_2 = \mathfrak{A}_3$.

In order to present the following results on absolutely free decompositions in a more compact form, it is helpful to extend slightly the notion of an absolutely free product over a given amalgamated subsystem. We agree to let the symbol $*$ denote absolutely free products, and the symbol $\underset{\mathfrak{C}}{*}$ absolutely free products over an amalgamated subsystem $\mathfrak{C}$. We shall say that a system $\mathfrak{A}$ is the *mixed free product* of its subsystems $\mathfrak{A}_i (i \in I)_k$ over the amalgamated subsystem $\mathfrak{C}$, in symbols,

$$\mathfrak{A} = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{A}_i \qquad (i \in I),$$

if $I = K \cup L$, each factor $\mathfrak{A}_i$ contains $\mathfrak{C}$ for $i \in K$, and $\mathfrak{A}_i \cap \mathfrak{C} = \varnothing$ for $i \in L$ and in addition if

$$\mathfrak{A} = (\prod \underset{\mathfrak{C}}{*} \mathfrak{A}_k) * (\prod * \mathfrak{A}_l) \qquad (k \in K,\ l \in L).$$

The basic property of mixed free products is given by

Corollary 7. *Mixed free products are associative, i.e. if* $\mathfrak{A} = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{A}_i$ $(i \in I)$, if $I = \bigcup I_\mu$ $(\mu \in M)$ *is a partition of* I *and* $\mathfrak{B}_\mu$ *is the subsystem*

generated in $\mathfrak{A}$ *by elements of the systems* $\mathfrak{A}_i$ $(i \in I_\mu)$, *then*

$$\mathfrak{B}_\mu = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{A}_i \qquad (i \in I_\mu),$$

$$\mathfrak{A} = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{B}_\mu \qquad (\mu \in M).$$

To prove this, one merely has to apply Theorem 3.

Analogously, from Theorem 4 and 5, it is easy to obtain

Corollary 8. *If* $\mathfrak{A} = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{A}_i$ $(i \in I)$ *and* $\mathfrak{B}$ *is a subsystem of* $\mathfrak{A}$ *either containing the amalgamated subsystem* $\mathfrak{C}$ *or else disjoint from it, then*

$$\mathfrak{B} = \prod \underset{\mathfrak{C}}{\circledast} (\mathfrak{A}_i \cap \mathfrak{B}) * \mathfrak{B},$$

where the factor $\mathfrak{B}$ *is either empty or is an absolutely free subsystem of* $\mathfrak{A}$ *disjoint from* $\mathfrak{C}$.

Indeed, grouping the factors $\mathfrak{A}_i$, we have

$$\mathfrak{A} = (\prod \underset{\mathfrak{C}}{*} \mathfrak{A}_k) * (\prod * \mathfrak{A}_l).$$

Applying Theorems 4 or 5, we obtain

$$\mathfrak{B} = (\mathfrak{B} \cap \prod \underset{\mathfrak{C}}{*} \mathfrak{A}_k) * (\mathfrak{B} \cap \prod * \mathfrak{A}_l) * \mathfrak{B}_1.$$

Applying them once again, we obtain the required decomposition for $\mathfrak{B}$.

We shall say that the decomposition

$$\mathfrak{A} = \prod \underset{\mathfrak{C}}{*} \mathfrak{A}_i \qquad (i \in I)$$

of a system $\mathfrak{A}$ is a *subdecomposition* (or *refinement*) of the decomposition $\mathfrak{A} = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{B}_\mu$ $(\mu \in M)$ if there exists a partition $I = \bigcup I_\mu$ $(\mu \in M)$ such that

$$\mathfrak{B}_\mu = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{A}_i \qquad (i \in I_\mu).$$

Theorem 9. *Any two mixed free decompositions of an arbitrary system* $\mathfrak{A}$ *over the same amalgamated subsystem* $\mathfrak{C}$ *have a common mixed free subdecomposition over* $\mathfrak{C}$.

Suppose

$$\mathfrak{A} = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{A}_i = \prod \underset{\mathfrak{C}}{\circledast} \mathfrak{B}_j \qquad (i \in I,\ j \in J) \tag{4}$$

are the given decompositions of $\mathfrak{A}$. By Corollary 8,

$$\mathfrak{A}_i = \mathfrak{B}_i * \prod \underset{\mathfrak{C}}{\circledast} (\mathfrak{A}_i \cap \mathfrak{B}_j), \quad \mathfrak{B}_j = \mathfrak{W}_j * \prod \underset{\mathfrak{C}}{\circledast} (\mathfrak{A}_i \cap \mathfrak{B}_j), \qquad (5)$$

where $\mathfrak{B}_i$ and $\mathfrak{W}_j$ are suitable absolutely free subsystems of $\mathfrak{A}$ or are empty sets. From the decompositions (5), we obtain

$$\mathfrak{A} = \prod_{i,j} \underset{\mathfrak{C}}{\circledast} (\mathfrak{A}_i \cap \mathfrak{B}_j) * \prod_i * \mathfrak{B}_i = \prod_{i,j} \underset{\mathfrak{C}}{\circledast} (\mathfrak{A}_i \cap \mathfrak{B}_j) * \prod_j * \mathfrak{W}_j.$$

According to Theorem 6, this implies that $\prod * \mathfrak{B}_i = \prod * \mathfrak{W}_j$. Absolutely free systems can be decomposed into absolutely free products of their absolutely free subsystems of rank 1 and this decomposition is unique. Therefore,

$$\prod * \mathfrak{B}_i = \prod * \mathfrak{W}_j = \prod * \mathfrak{F}_k \qquad (k \in K),$$

where the $\mathfrak{F}_k$ are absolutely free systems of rank 1. From the uniqueness of the decomposition, it follows that for suitable partitions

$$K = \cup K_i = \cup K_j'$$

we have

$$\mathfrak{B}_i = \prod * \mathfrak{F}_k \quad (k \in K_i), \qquad \mathfrak{W}_j = \prod * \mathfrak{F}_k \quad (k \in K_j').$$

Therefore the decomposition

$$\mathfrak{A} = \prod_{i,j} \underset{\mathfrak{C}}{\circledast} (\mathfrak{A}_i \cap \mathfrak{B}_j) * \prod_k * \mathfrak{F}_k$$

is the desired common subdecomposition of the decompositions (4).

A system $\mathfrak{A}$ will be called *mixed indecomposable* over $\mathfrak{C}$ if it can not be represented as a mixed free product over $\mathfrak{C}$ of two of its subsystem distinct from $\mathfrak{A}$. From Theorem 9, we immediately obtain

Corollary 10. *If a system $\mathfrak{A}$ can be represented in the form of a mixed free product over $\mathfrak{C} \subset \mathfrak{A}$ of its mixed indecomposable systems over $\mathfrak{C}$, then such a representation is unique.*

One can construct (see Problems and Complements below, Item 8) a system decomposable into an absolutely free product of any finite number of its (nonempty) subsystems but which is not decomposable into an absolutely free product of subsystems which are themselves not decomposable into an absolutely free product.

Problems and Complements

1. Let. $\mathfrak{K}$ be a replete class. Every system definable in $\mathfrak{K}$ by an infinite collection of generating symbols and a finite collection of defining relations is a $\mathfrak{K}$-free composition of a finitely presented system and a $\mathfrak{K}$-free system of infinite rank.

2. The *basis rank* of a system $\mathfrak{A}$ is the minimal cardinality of a generating set for the system. Show that the basis rank of an absolutely free composition of systems is equal to the sum of the basis ranks of the individual components.

3. In the class of commutative semigroups, the basis rank of a free composition may be less than the sum of the basis ranks of the component semigroups.

4. An element $a \in \mathfrak{A}$ is called an *algebraic constant* in the system $\mathfrak{A} = \langle A, \Omega \rangle$ if there exists a term $f(x)$ of signature Ω, all of whose values in $\mathfrak{A}$ are equal to a. Give an example of an algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{B}$ such that algebraic constants exist in $\mathfrak{B}$ but not in $\mathfrak{A}$.

5. If a self-independent element a of a system $\mathfrak{A}$ is an algebraic constant in $\mathfrak{A}$, then $\mathfrak{A}$ consists of a single element.

6. If S is an independent set of elements of $\mathfrak{A}$, then for any $U \subseteq S$ and $V \subseteq S$, we have $\overline{U} \cap \overline{V} = \overline{U \cap V}$, where $\overline{X}$ denotes the subsystem generated in $\mathfrak{A}$ by set $X \subseteq \mathfrak{A}$ (Marczewski [50]).

7. Let $X = \{x_0, x_1, \ldots\}$ be a set of indeterminates of cardinality $\mathfrak{m} \geqq 1$. Denote by F the collection of linear forms $k_1 x_{i_1} + \cdots + k_r x_{i_r}$ $(x_{i_1}, \ldots, x_{i_r} \in X)$ with integer coefficients. The group $\mathfrak{F}_\mathfrak{m} = \langle F, +, - \rangle$ is a free group of rank $\mathfrak{m}$ in the variety of all abelian groups.

8. Let $\mathfrak{G}$ be the groupoid defined in the class of all groupoids by generators a_i and b_i and relations $a_{i+1} b_{i+1} = a_i$ $(i = 1, 2, \ldots)$. Denote by $\mathfrak{G}_n$ the subgroupoid of $\mathfrak{G}$ generated by the elements a_{n+i} and b_{n+i} $(i = 1, 2, \ldots)$ and by $\mathfrak{H}_n$ the subgroupoid generated by the element b_n $(n = 1, 2, \ldots)$. Then

$$\mathfrak{G} = \mathfrak{H}_1 * \ldots * \mathfrak{H}_n * \mathfrak{G}_n \qquad (n = 1, 2, \ldots).$$

However, the groupoid $\mathfrak{G}$ is not decomposable into an absolutely free product of subgroupoids which are themselves not decomposable into an absolutely free product.

9. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two varieties of algebras of signature Ω_1 and Ω_2 defined by the systems of identities $\mathfrak{S}_1$ and $\mathfrak{S}_2$; suppose Ω_1 and Ω_2 intersect in the 0-ary operator 0, and $\mathfrak{S}_1$ and $\mathfrak{S}_2$ contain the identities $00 \ldots 0\omega = 0$ for every ω in Ω_1 and Ω_2, respectively. Denote by $\mathfrak{M}$ the variety of algebras of signature $\Omega_1 \cup \Omega_2$ defined by the system of identities $\mathfrak{S}_1 \cup \mathfrak{S}_2$. If for each $i = 1,2$ the subalgebras of the $\mathfrak{M}$-free algebras are $\mathfrak{M}_i$-free, then the subalgebras of $\mathfrak{M}$-free algebras are $\mathfrak{M}$-free (Baranovič [6]).

Chapter VI

Varieties

Varieties of algebraic systems are classes which can be characterized by collections of identities and the theory of varieties is equivalent to the theory of systems of identities. Although identities are the simplest closed logical expressions, the language of identities is still rich enough to express many delicate properties of systems and their classes. It is mainly for this reason that the theory of varieties holds a special place in general algebra.

The deepest results and problems in the theory of varieties are at the present time connected with the study of the subvarieties of various specific varieties, for example, varieties of groups or rings. In this chapter we shall only consider the properties of general varieties; although not as technically complex, they will have broad application.

13. General Properties

13.1 Structural Characteristics. At the foundations of the theory of varieties is the following theorem, proved for varieties of algebras by G. Birkhoff in 1935.

Theorem 1. (Birkhoff [7]). *A nonempty class $\mathfrak{K}$ of algebraic systems is a variety if and only if*

a) *the direct product of an arbitrary sequence of $\mathfrak{K}$-systems is a $\mathfrak{K}$-system,*

b) *any subsystem of an arbitrary $\mathfrak{K}$-system is a $\mathfrak{K}$-system.*

c) *any homomorphic image of an arbitrary $\mathfrak{K}$-system is a $\mathfrak{K}$-system.*

In other words, it is necessary and sufficient that $\mathfrak{K}$ be hereditary, multiplicatively closed and homomorphically closed.

Necessity. This part is obvious since the identities which characterize a variety are stable under the operations in a), b), and c).

Sufficiency. The trivial system of given signature Ω is the homomorphic image of any system of this signature and hence by c), the trivial system certainly belongs to $\mathfrak{K}$. From properties a) and b), we conclude that $\mathfrak{K}$ is replete and consequently, $\mathfrak{K}$ has free systems of arbitrary rank.

Let $I\mathfrak{K}$ denote the collection of all identities of a given signature which hold in all $\mathfrak{K}$-systems. We must prove that $KI\mathfrak{K} = \mathfrak{K}$. Since the inclusion $\mathfrak{K} \subseteq KI\mathfrak{K}$ is trivial, it is sufficient to prove $KI\mathfrak{K} \subseteq \mathfrak{K}$.

Suppose $\mathfrak{A} \in KI\mathfrak{K}$ and $|\mathfrak{A}| = \mathfrak{m}$. By the above remark, $\mathfrak{K}$ has a free system $\mathfrak{F}$ whose free basis is of cardinality $\mathfrak{m}$. Suppose $z_a \to a$ ($z_a \in \mathfrak{F}$, $a \in \mathfrak{A}$) is some mapping of the free basis of $\mathfrak{F}$ onto $\mathfrak{A}$. We define a relation $\zeta \subseteq \mathfrak{F} \times \mathfrak{A}$ by putting $\langle u, b\rangle \in \zeta$ if there exists a term $f(x_1, \ldots, x_n)$ of signature Ω for which

$$u = f(z_{a_1}, \ldots, z_{a_n}), \qquad b = f(a_1, \ldots, a_n)$$

for suitable $a_1, \ldots, a_n$ in $\mathfrak{A}$. Since the system $\mathfrak{F}$ is generated by the elements z_a, the first component of the relation ζ coincides with $\mathfrak{F}$. Let us show that ζ is a homomorphism of $\mathfrak{F}$ onto $\mathfrak{A}$. Suppose that for terms $f_1, \ldots, f_s$ and distinct elements $z_{a_1}, \ldots, z_{a_n}$ of the free basis of $\mathfrak{F}$,

$$P\big(f_1(z_{a_1}, \ldots, z_{a_n}), \ldots, f_s(z_{a_1}, \ldots, z_{a_n})\big) = \mathsf{T} \qquad (P \in \{\Omega, =\}). \tag{1}$$

Since the $z_{a_1}, \ldots, z_{a_n}$ are $\mathfrak{K}$-free elements, it follows from (1) that the identity

$$P\big(f_1(x_1, \ldots, x_n), \ldots, f_s(z_1, \ldots, x_n)\big) = \mathsf{T} \tag{2}$$

holds in $\mathfrak{K}$.

By hypothesis, the system $\mathfrak{A}$ satisfies every identity which holds in the class $\mathfrak{K}$ and hence identity (2) holds in $\mathfrak{A}$. Substituting the elements $a_1, \ldots, a_n$ for $x_1, \ldots, x_n$ in this identity, we conclude that the relation

$$P\big(f_1(a_1, \ldots, a_n), \ldots, f_s(a_1, \ldots, a_n)\big)$$

is true in $\mathfrak{A}$.

Thus, for any $u_1, \ldots, u_s$ in $\mathfrak{F}$ and $b_1, \ldots, b_s$ in $\mathfrak{A}$, we have

$$P(u_1, \ldots, u_s) \,\&\, u_1 \zeta b_1 \,\&\, \ldots \,\&\, u_s \zeta b_s \Rightarrow P(b_1, \ldots, b_s)$$
$$(P \in \{\Omega, =\}),$$

and so ζ is a homomorphism of $\mathfrak{F}$ onto $\mathfrak{A}$.

Since $\mathfrak{F} \in \mathfrak{K}$ and $\mathfrak{A}$ is a homomorphic image of $\mathfrak{F}$, by c), $\mathfrak{A} \in \mathfrak{K}$, as required.

For any class $\mathfrak{K}$, we define

$$\hat{\mathfrak{K}} = KI\mathfrak{K},$$

i.e. $\hat{\mathfrak{K}}$ denotes the *minimal variety* containing the class $\mathfrak{K}$.

Theorem 2. *For any class* $\mathfrak{K}$,

$$\hat{\mathfrak{K}} = HS\prod \mathfrak{K}, \tag{3}$$

where $H\mathfrak{L}$ denotes the class consisting of the homomorphic images of $\mathfrak{L}$-systems.

The inclusions $\mathfrak{K} \subseteq HS\prod \mathfrak{K} \subseteq \hat{\mathfrak{K}}$ are evident. Therefore, we need only prove that the class $HS\prod \mathfrak{K}$ is a variety, or according to the Birkhoff Theorem, that it has properties a)—c).

Property c) is clear for $HS\prod \mathfrak{K}$ since the homomorphic image of a homomorphic image of a system is a homomorphic image of that system. To verify properties a) and b), we set $S\prod \mathfrak{K} = \mathfrak{L}$ so that $HS\prod \mathfrak{K} = H\mathfrak{L}$. By Section 2.5, the class $\mathfrak{L}$ is closed under direct products and formation of subsystems. Suppose $\mathfrak{A} \in H\mathfrak{L}$ and $\mathfrak{A}_1$ is any subsystem of $\mathfrak{A}$. By hypothesis, there exists an epimorphism of the form

$$\alpha\colon \mathfrak{B} \to \mathfrak{A} \qquad (\mathfrak{B} \in \mathfrak{L}).$$

Let $\mathfrak{B}_1$ denote the complete pre-image of $\mathfrak{A}_1$ in $\mathfrak{B}$. Since $\mathfrak{B}_1$ is a subsystem of an $\mathfrak{L}$-system $\mathfrak{B}$, we have $\mathfrak{B}_1 \in \mathfrak{L}$, i.e. $\mathfrak{A}_1$ is the homomorphic image of an $\mathfrak{L}$-system and hence $\mathfrak{A}_1 \in H\mathfrak{L}$. This proves property b) for the class $H\mathfrak{L}$.

Similarly, let $\mathfrak{A}_i$ $(i \in I)$ be a sequence of $H\mathfrak{L}$-systems. By hypothesis, there exist epimorphisms

$$\alpha_i\colon \mathfrak{B}_i \to \mathfrak{A} \qquad (i \in I),$$

where the $\mathfrak{B}_i$ are suitable $\mathfrak{L}$-systems. According to Section 2.5, the existence of the epimorphisms α_i implies the existence of the epimorphism

$$\alpha\colon \prod \mathfrak{B}_i \to \prod \mathfrak{A}_i.$$

Since the system $\prod \mathfrak{B}_i$ $(i \in I)$ belongs to the class $\mathfrak{L}$, the product $\prod \mathfrak{A}_i$ $(i \in I)$ belongs to the class $H\mathfrak{L}$, i.e. $H\mathfrak{L}$ has property a).

It is clear that *every variety is completely determined by its free systems.* Indeed, every system of a variety is a homomorphic image of a free system of suitable rank and all homomorphic images of free systems belong to the variety. However, an even stronger assertion is true. We formulate it as the first part of the following theorem.

Theorem 3. *Every variety is completely determined by its free system of countable rank. For any class of systems $\mathfrak{K}$, the free systems in the minimal replete class $S\prod \mathfrak{K}_e$ are also free systems in the minimal variety $\hat{\mathfrak{K}}$.*

It is clear that if a class $\mathfrak{L}$ has a free system $\mathfrak{F}_\omega$ of countable rank, then $I\mathfrak{L} = I\mathfrak{F}_\omega$. Every nontrivial variety $\mathfrak{K}$ has the free system $\mathfrak{F}_\omega$ and moreover, $KI\mathfrak{K} = \mathfrak{K}$. Therefore, for any variety $\mathfrak{K}$ we have the formula

$$\mathfrak{K} = KI\mathfrak{F}_\omega, \tag{4}$$

which, in particular, proves the first assertion of Theorem 3.

To prove the second assertion, we merely have to recall Theorem 3 of Section 12.2 which says that a system which is free relative to the class $S\prod \mathfrak{K}_e$ is also free relative to the class $HS\prod \mathfrak{K} = \hat{\mathfrak{K}}$.

As an illustration, consider the quasivariety S_e of semigroups embeddable in a group (see Section 11.1). The free semigroup (in the class of all semigroups) with generators z_i ($i \in I$) is the semigroup of nonempty words in the alphabet $\{z_i\}$ under the usual operation of concatenation of words. This semigroup is a multiplicative subsemigroup of the free group and hence belongs to the class S_e. Thus, the quasivariety S_e and the class of all semigroups have the same free semigroups. If S_e were a variety, then it would coincide with the class of all semigroups, which is certainly not the case. Therefore, the quasivariety S_e is not a variety.

We pose the question: *Under what conditions will an arbitrary given system* $\mathfrak{A}$ *be free in some class* $\mathfrak{K}$?

A system $\mathfrak{A}$ free in $\mathfrak{K}$ is also free in the variety $\hat{\mathfrak{K}}$ (by Section 12.2 and Theorem 3). Let $\hat{\mathfrak{A}}$ denote the minimal variety containing the system $\mathfrak{A}$. Since $\mathfrak{A} \in \mathfrak{K}$, it follows that $\hat{\mathfrak{A}} \subseteq \hat{\mathfrak{K}}$, and hence $\mathfrak{A}$ is a free system in $\hat{\mathfrak{A}}$. According to the same Theorem 3, in order that a system $\mathfrak{A}$ be free in the variety $\hat{\mathfrak{A}}$, it is sufficient (and clearly, necessary) that $\mathfrak{A}$ be free relative to itself, i.e. that $\mathfrak{A}$ contain an independent set of generating elements. We have obtained

Corollary 4. *A system* $\mathfrak{A}$ *is free in some class if and only if it has an independent generating set of elements. In this case,* $\mathfrak{A}$ *is free in the variety* $\hat{\mathfrak{A}}$.

A system $\mathfrak{A}$ of signature Ω is said to be *equationally contained* in system $\mathfrak{B}$ if $\mathfrak{B}$ has signature Ω and every identity which is true in $\mathfrak{A}$ is true in $\mathfrak{B}$, i.e. if $I\mathfrak{A} \subseteq I\mathfrak{B}$. From the obvious equivalences

$$I\mathfrak{A} \subseteq I\mathfrak{B} \Leftrightarrow KI\mathfrak{A} \supseteq KI\mathfrak{B} \Leftrightarrow \mathfrak{B} \in \hat{\mathfrak{A}}$$

and formula (3), we conclude that *a system* $\mathfrak{B}$ *equationally contains system* $\mathfrak{A}$ *if and only if* $\mathfrak{B}$ *is a homomorphic image of a subsystem of a suitable direct power of* $\mathfrak{A}$.

Systems $\mathfrak{A}$ and $\mathfrak{B}$ are called *equationally equivalent* (or I-equivalent) if each equationally contains the other, i.e. if $I\mathfrak{A} = I\mathfrak{B}$. It is clear that the last condition is equivalent to $\hat{\mathfrak{A}} = \hat{\mathfrak{B}}$. Isomorphic systems are obviously equationally equivalent. In general, the converse does not hold. From the preceding discussion, we obtain the following simple test for equational equivalence.

Corollary 5. *If* $\mathfrak{A}$ *is a subsystem of a direct power of the system* $\mathfrak{B}$ *and* $\mathfrak{B}$ *is a subsystem or homomorphic image of* $\mathfrak{A}$, *then* $\hat{\mathfrak{A}} = \hat{\mathfrak{B}}$.

For instance, $\mathfrak{A}$ and any direct power of $\mathfrak{A}$ are equationally equivalent.

Consider also the following example. Let $\mathfrak{J} = \langle \{0, 1\}, \cdot \rangle$ be the semigroup in which

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1,$$

and suppose $\mathfrak{A}$ is an arbitrary commutative idempotent semigroup containing at least two elements $a \neq b$. If $ab = a$ or $ba = b$, then the elements a and b have one of the following multiplication tables in $\mathfrak{A}$:

	a	b
a	a	a
b	a	b,

	a	b
a	a	b
b	b	b

In both cases, $\mathfrak{A}$ contains a subsemigroup $\{a, b\}$ isomorphic to $\mathfrak{J}$.

Suppose $ab \neq a$ and $ab \neq b$. Then the elements a and ab form a subsemigroup isomorphic to $\mathfrak{J}$ since they multiply in $\mathfrak{A}$ according to the table

	a	ab
a	a	ab
ab	ab	ab

On the other hand, Section 3.1 showed that every commutative idempotent semigroup is isomorphically embeddable in a suitable direct power of the semigroup $\mathfrak{J}$.

By Corollary 5 we can assert that *any two nontrivial commutative idempotent semigroups are equationally equivalent* since each of them is equationally equivalent to the semigroup $\mathfrak{J}$.

Considering quasi-identities instead of identities, we arrive at the notion of quasi-equational equivalence (or, Q-equivalence) of algebraic systems. We have

Corollary 6. *If $\mathfrak{A}$ is a subsystem of a filtered power of a system $\mathfrak{B}$ and $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are Q-equivalent.*

The above considerations show for instance, that any two nontrivial commutative idempotent semigroups are in fact Q-equivalent.

13.2 Ranks of Varieties. The *rank* of an identity or quasi-identity is the number of distinct free object variables which occur in it. For example, the ranks of the formulas

$$x^2 \leqq x^3, \qquad xy = yx, \qquad x(yz) = (xy)z$$

are 1, 2 and 3, respectively. In this connection, we define the *axiomatic rank* $r_a(\mathfrak{K})$ of a variety $\mathfrak{K}$ as the least natural number r such that $\mathfrak{K}$ can be characterized by a collection of identities whose ranks do not exceed r.

If there is no natural number r with this property, then we shall say that the axiomatic rank of $\mathfrak{K}$ is infinite (in symbols, $r_a(\mathfrak{K}) = \infty$).

Let $I_n\mathfrak{K}$ denote the collection of all identities of rank $\leqq n$ which are true in the class $\mathfrak{K}$, and let $K\mathfrak{S}$ as usual denote the class of all algebraic systems which satisfy all the formulas of the collection $\mathfrak{S}$. Set $\mathfrak{K}^n = KI_n\mathfrak{K}$. Then to each class of systems $\mathfrak{K}$, there is an associated sequence of varieties $\mathfrak{K}^n$ such that

$$\mathfrak{K}^1 \supseteq \mathfrak{K}^2 \supseteq \ldots \supseteq \mathfrak{K}^n \supseteq \ldots \supseteq \mathfrak{K}. \tag{1}$$

It is clear that $\mathfrak{K} = \bigcap \mathfrak{K}^n$ if and only if $\mathfrak{K}$ is a variety. We especially take note of

Corollary 1. *An algebraic system $\mathfrak{A}$ (with signature that of $\mathfrak{K}$) belongs to the variety $\mathfrak{K}^n$ if and only if each of its subsystems generated by at most n elements, belongs to $\mathfrak{K}^n$.*

Indeed, the condition $\mathfrak{A} \in \mathfrak{K}^n$ is equivalent to the fact that all the identities of $I_n\mathfrak{K}$ hold in $\mathfrak{A}$. Each of these identities has the form

$$(\forall x_1 \ldots x_n) P\big(f_1(x, \ldots, x_n), \ldots, f_s(x_1, \ldots, x_n)\big)$$
$$(P \in \{\Omega, =\}).$$

To verify that it holds in $\mathfrak{A}$, we must take arbitrary elements $x_1, \ldots, x_n$, compute $f_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, s)$ and determine whether the relation P holds for the resulting values $f_1, \ldots, f_s$. But the evaluation of $f_1, \ldots, f_s$ takes place in a subsystem generated by the n elements $x_1, \ldots, x_n$ and hence, the verification need be made only in such subsystems.

It was shown above that for every variety $\mathfrak{K}$,

$$KI\mathfrak{F}_\omega(\mathfrak{K}) = \mathfrak{K}.$$

However, it may happen that $\mathfrak{K}$ is completely determined by its free system $\mathfrak{F}_n(\mathfrak{K})$ of finite rank n. The least n such that

$$KI\mathfrak{F}_n(\mathfrak{K}) = \mathfrak{K}$$

is called the *basis rank* of $\mathfrak{K}$ and is denoted by $r_b = r_b(\mathfrak{K})$. Introducing the notation $KI\mathfrak{F}_i(\mathfrak{K}) = \mathfrak{K}_i$, we obtain as a complement to the chain (1) the sequence of varieties $\mathfrak{K}_i$:

$$\mathfrak{K}_1 \subseteq \mathfrak{K}_2 \subseteq \ldots \subseteq \mathfrak{K}_n \subseteq \ldots \subseteq \mathfrak{K}, \tag{2}$$

and $\mathfrak{K} = \bigcup \mathfrak{K}_n$ if $\mathfrak{K}$ is a variety.

In computing the basis ranks of specific varieties, we sometimes find it useful to have

Corollary 2. *If* $\mathfrak{F}_n(\mathfrak{K})$ *contains a subsystem isomorphic to* $\mathfrak{F}_{n+1}(\mathfrak{K})$, *then* $\mathfrak{K}_n = \mathfrak{K}_{n+1}$. *In particular, if* $\mathfrak{K}$ *is a variety and* $\mathfrak{F}_n(\mathfrak{K})$ *contains a subsystem isomorphic to* $\mathfrak{F}_{n+i}(\mathfrak{K})$ *for each finite* i, *then* $\mathfrak{K}_n = \mathfrak{K}$ *and* $r_b(\mathfrak{K}) \leqq n$.

Since every identity which holds in $\mathfrak{F}_n$ also holds in any subsystem of $\mathfrak{F}_n$, we have $I\mathfrak{F}_n \subseteq I\mathfrak{F}_{n+i}$; consequently, $KI\mathfrak{F}_n \supseteq KI\mathfrak{F}_{n+i}$. This together with the inclusions (2) gives $\mathfrak{K} = \mathfrak{K}_{n+i}$. If the latter equation holds for any i, then from the relation $\mathfrak{K} = \bigcup \mathfrak{K}_{n+i}$ we obtain $\mathfrak{K} = \mathfrak{K}_n$.

The *basis rank* $r_b\mathfrak{A}$ of a given system $\mathfrak{A}$ was defined as the least of the cardinalities of the generating sets for $\mathfrak{A}$. What is the relation between the basis rank of $\mathfrak{A}$ and the basis rank of the minimal variety $\hat{\mathfrak{A}}$ generated by the system $\mathfrak{A}$?

Theorem 3. *For any system* $\mathfrak{A}$,

$$r_b\hat{\mathfrak{A}} \leqq r_b\mathfrak{A}. \tag{3}$$

The basis rank of an arbitrary variety $\mathfrak{K}$ *coincides with the least of the basis ranks of the* $\mathfrak{K}$*-systems which generate the variety* $\mathfrak{K}$.

Suppose that $r_b\mathfrak{A} = n$ and $\mathfrak{F}_n$ is a free system of rank n in the variety $\hat{\mathfrak{A}}$. Then there is an epimorphism $\mathfrak{F}_n \to \mathfrak{A}$ and hence

$$I\mathfrak{F}_n \subseteq I\mathfrak{A}, \quad KI\mathfrak{F}_n \supseteq KI\mathfrak{A},$$

i.e. $\hat{\mathfrak{F}}_n \supseteq \hat{\mathfrak{A}}$. But $\mathfrak{F}_n \in \hat{\mathfrak{A}}$, thus $\hat{\mathfrak{F}}_n = \hat{\mathfrak{A}}$ and consequently $r_b\hat{\mathfrak{A}} \leqq n$. Put $r_b\mathfrak{K} = n$. By definition, this means that

$$\mathfrak{K} = \hat{\mathfrak{F}}_n, \quad \mathfrak{K} \neq \hat{\mathfrak{F}}_i \quad (i < n).$$

With the already established inequality (3), we obtain

$$n = r_b\hat{\mathfrak{F}}_n \leqq r_b\mathfrak{F}_n \leqq n,$$

from which $r_b\mathfrak{F}_n = n$. Thus, every variety $\mathfrak{K}$ of basis rank n is certainly generated by a $\mathfrak{K}$-system of basis rank n. A system of smaller basis rank can not generate the variety $\mathfrak{K}$ since from inequality (3) we obtain

$$r_b\mathfrak{K} = r_b\hat{\mathfrak{B}} \leqq r_b\mathfrak{B}$$

for any system $\mathfrak{B}$ which generates the variety $\mathfrak{K}$. This proves Theorem 3.

In the above proof, we encountered the equation $r_b\mathfrak{F}_n = n$, where $\mathfrak{F}_n$ is free in $\mathfrak{K}$ of rank n. A question arising is whether this equation holds in any variety? The answer is no. A counterexample is given by the free algebras $\mathfrak{F}_n$ in the variety $\mathfrak{S}_{1,2}$ (Section 12.2). We saw that in this variety $\mathfrak{F}_1 \cong \mathfrak{F}_n$ for any finite n and hence

$$r_b\mathfrak{F}_n = r_b\mathfrak{F}_1 = 1 \quad (n = 2, 3, \ldots).$$

Let us say that a system $\mathfrak{F}_n$ of finite rank n which is free in itself, has property $\mathscr{B}$ if an arbitrary set $a_1, \ldots, a_n$ of (not necessarily distinct) elements of $\mathfrak{F}_n$ which generates $\mathfrak{F}_n$ is independent in $\mathfrak{F}_n$; that is, any mapping $a_i \to b_i \in \mathfrak{F}_n$ $(i = 1, \ldots, n)$ can be extended to a homomorphism of $\mathfrak{F}_n$ into itself.

Of course, if $|\mathfrak{F}_n| \geqq 2$, then

$$\mathscr{B}(\mathfrak{F}_n) \Rightarrow r_b\mathfrak{F}_n = n.$$

Indeed, if $\mathfrak{F}_n$ has property $\mathscr{B}$ and the sequence $a_1, \ldots, a_m$ $(m < n)$ generates $\mathfrak{F}_n$, then certainly the nonfree sequence $a_1, \ldots, a_m, a_m, \ldots, a_m$ also generates $\mathfrak{F}_n$.

Which systems $\mathfrak{F}_n$ have property $\mathscr{B}$? A preliminary answer to this question is given by

Theorem 4. *In a free system $\mathfrak{F}_n$ of finite rank n, any generating collection of n elements is free if and only if any epimorphism of $\mathfrak{F}_n$ onto itself is an isomorphism.*

Suppose $\varphi\colon \mathfrak{F}_n \to \mathfrak{F}_n$ is an epimorphism which is not an isomorphism and that $\{v_1, \ldots, v_n\}$ is a free generating set of elements of $\mathfrak{F}_n$. Then $\{v_1\varphi, \ldots, v_n\varphi\}$ also generates $\mathfrak{F}_n$. If the latter set were free, then there would exist an isomorphism $\varphi\colon \mathfrak{F}_n \to \mathfrak{F}_n$ taking $v_1, \ldots, v_n$ into $v_1\varphi$, $\ldots, v_n\varphi$, respectively. Since the homomorphisms φ and ψ coincide on the generating set $\{v_1, \ldots, v_n\}$, they must coincide also on $\mathfrak{F}_n$, i.e. $\varphi = \psi$, which contradicts the assumption.

Conversely, suppose that any set of n elements which generates $\mathfrak{F}_n$ is free in $\mathfrak{F}_n$, and suppose that $\varphi\colon \mathfrak{F}_n \to \mathfrak{F}_n$ is some epimorphism. Then the elements $v_1\varphi, \ldots, v_n\varphi$ generate $\mathfrak{F}_n$ and hence are free in $\mathfrak{F}_n$. Therefore, there must exist an isomorphism $\psi\colon \mathfrak{F}_n \to \mathfrak{F}_n$ taking $v_1, \ldots, v_n$ into $v_1\varphi, \ldots, v_n\varphi$, respectively. Since the homomorphisms φ and ψ coincide on the generators $v_1, \ldots, v_n$, we have $\varphi = \psi$, as required.

As an example, we shall compute r_a and r_b for the variety of all semigroups. This variety is defined by the single identity $xy \cdot z = x \cdot yz$ (Section 3.1). Therefore $r_a \leqq 3$.

On the other hand, the Cayley numbers form as we know, a groupoid $\mathfrak{G}$ under multiplication in which any two elements generate a semigroup (Section 4.3). If the variety of all semigroups had axiomatic rank 2, then by Corollary 1, the groupoid $\mathfrak{G}$ would also be a semigroup. Since $\mathfrak{G}$ is not associative, $r_a = 3$.

To compute r_b, we note that in the variety of all semigroups, the free semigroup of rank 1 satisfies the identity $xy = yx$, which does not hold in every semigroup. Consequently, $r_b \geqq 2$. On the other hand, the free semigroup of rank 2 contains a subsemigroup isomorphic to the free semigroup of countable rank. By Corollary 2, $r_b \leqq 2$ and hence $r_b = 2$.

Analogously, for the variety of all groups, we have $r_a = 3$, $r_b = 2$.

For certain varieties, the question as to what values r_a and r_b have on subvarieties seems to be difficult. For example, it is not yet known whether there are varieties of groups for which $r_a = \infty$.*)

By analogy, the *axiomatic quasirank* $q_a = q_a(\mathfrak{K})$ of a quasivariety $\mathfrak{K}$ is defined as the least number n such that $\mathfrak{K}$ can be characterized by a collection of quasi-identities each of which contains no more than n free variables. If there is no such number, then we set $q_a = \infty$.

Denote by $Q\mathfrak{K}$ the collection of all quasi-identities and by $Q_n\mathfrak{K}$ the collection of all quasi-identities of rank n (with signature that of $\mathfrak{K}$) which hold in the arbitrary class $\mathfrak{K}$. Putting

$$\mathfrak{K}^{(n)} = KQ_n(\mathfrak{K}) \qquad (n = 1, 2, \ldots),$$

we obtain a chain of quasivarieties

$$\mathfrak{K}^{(1)} \supseteq \mathfrak{K}^{(2)} \supseteq \cdots \supseteq \mathfrak{K}^{(n)} \supseteq \cdots \supseteq \mathfrak{K}.$$

The relation $\mathfrak{K} = \bigcap \mathfrak{K}^{(n)}$ is equivalent to the assertion that $\mathfrak{K}$ is a quasivariety. Just as before, we have

Corollary 5. *A system $\mathfrak{A}$ belongs to the quasivariety $\mathfrak{K}^{(n)}$ if and only if this quasivariety contains any subsystem of $\mathfrak{A}$ generated by any n elements of $\mathfrak{A}$.*

In every quasivariety $\mathfrak{K}$, there exist free systems $\mathfrak{F}_n(\mathfrak{K})$ of arbitrary rank n. Putting

$$\mathfrak{K}_{(n)} = KQ\mathfrak{F}_n(\mathfrak{K}) \qquad (n = 1, 2, \ldots),$$

we obtain an ascending chain of quasivarieties

$$\mathfrak{K}_{(1)} \subseteq \mathfrak{K}_{(2)} \subseteq \cdots \subseteq \mathfrak{K}_{(n)} \subseteq \cdots \subseteq \mathfrak{K}.$$

If for some n, $\mathfrak{K}_{(n-1)} \neq \mathfrak{K}_{(n)} = \mathfrak{K}$, then n is called the *basis quasirank* of $\mathfrak{K}$ and is denoted by $q_b = q_b(\mathfrak{K})$. If for any natural number n, $\mathfrak{K}_{(1)} \neq \mathfrak{K}$ but $\bigcup \mathfrak{K}_{(n)} = \mathfrak{K}$, then we set $q_b = \infty$. It may happen that $\bigcup \mathfrak{K}_{(n)} \neq \mathfrak{K}$. Then the quasivariety $\mathfrak{K}$ is called *inexact* and its basis quasirank is undefined.

Consider, for example, the quasivariety $\mathfrak{S}_c$ of all cancellation semigroups. It is defined by the quasi-identities

$$x \cdot yz = xy \cdot z, \qquad xz = yz \to x = y, \qquad zx = zy \to x = y,$$

---

*) *Note added in proof.* The existence of a variety of groups for which $r_a = \infty$ has been proved: A. Ju. Ol'šanskiĭ (Izv. Akad. Nauk SSSR, Ser. Mat., 1970, 34, NO. 2, pp. 316–384), M. R. Vaughan-Lee (Bull. London Math. Soc., 1970, 2, NO. 6, pp. 280–286) and S. I. Adjan (Dokl. Akad. Nauk SSSR, 1970, 190, NO. 3, pp. 499–501) [*Ed*].

and therefore, $q_a \leqq 3$. Take the system $\mathfrak{A}$ to be collection of non-null Cayley numbers with the operation of multiplication. Applying Corollary 5, we see that $q_a > 2$. Thus, $q_a(\mathfrak{S}_c) = 3$. The free semigroup $\mathfrak{F}_1(\mathfrak{S}_c)$ is commutative and therefore $q_b > 1$. On the other hand, $\mathfrak{F}_2(\mathfrak{S}_c)$ contains free subsemigroups of any finite or countable rank. Therefore either $q_b = 2$, or the quasivariety $\mathfrak{S}_c$ is inexact. We show that the latter is true. Indeed, it is easy to check that the quasi-identity

$$x^2y^2 = z^2 \to xy = yx$$

holds in $\mathfrak{F}_2(\mathfrak{S}_c)$ but does not hold, for example, in the symmetric group $\mathfrak{C}_3$, which (like any group) is a cancellation semigroup under multiplication.

Examples of exact quasivarieties are the quasivarieties $\mathfrak{R}_{(n)}$. In particular, for $\mathfrak{S}_{c(2)}$, we have $q_a = 3$ and $q_b = 2$.

13.3 Varieties of Unoids. An algebra whose signature consists of m unary operator symbols is called an m-*unoid*. The collection of varieties of m-unoids for $m \geqq 2$ is quite rich and yet, at the same time, the study of its structure can be advanced considerably. Fundamental to this is the correspondence between varieties of m-unoids and semigroups with m distinguished generators.

Let us consider m-unoids of signature $\Omega = \{f_i \mid i \in I\}$, where $|I| = m$. Identities of this signature can have only the following four forms:

a) $f_{i_1} \dots f_{i_k}(x) = f_{j_1} \dots f_{j_l}(x)$,

b) $f_{i_1} \dots f_{i_k}(x) = x$,

c) $f_{i_1} \dots f_{i_k}(x) = f_{j_1} \dots f_{j_l}(y)$,

d) $f_{i_1} \dots f_{i_k}(x) = y$.

An identity of the form d) is equivalent to the identity $x = y$, which corresponds to the trivial algebra. Below we shall consider only those varieties of m-unoids which can be characterized by identities of the form a). They will be called varieties of m-unoids of the first kind.

Thus, suppose we are given a collection of identities

$$f_{i_{\lambda 1}} \dots f_{i_{\lambda s_\lambda}} = f_{j_{\lambda 1}} \dots f_{j_{\lambda t_\lambda}} \qquad (f_i, f_j \in \Omega). \tag{1}$$

This collection determines a certain variety $\mathfrak{R}$ of m-unoids. With the collection (1), we associate a set of defining symbols $V = \{v_i \mid i \in I\}$ and the following collection of defining relations in the class of semigroups:

$$v_{i_{\lambda 1}} \dots v_{i_{\lambda s_\lambda}} = v_{j_{\lambda 1}} \dots v_{j_{\lambda t_\lambda}}. \tag{2}$$

The symbols v_i and relations (2) define a semigroup $\mathfrak{A}$. We may assume that $\mathfrak{A}$ contains the elements v_i which generate it and are connected by the relations (2). Will the semigroup $\mathfrak{A}$ change if the variety $\mathfrak{K}$ is specified not by the system of identities (1) but by some other equivalent system? The negative answer is easily obtained from the following basic assertion.

Theorem 1. *An arbitrary identity*

$$f_{i_1} \dots f_{i_s}(x) = f_{j_1} \dots f_{j_t}(x) \tag{3}$$

follows from the collection (1) *if and only if the corresponding relation*

$$v_{i_1} \dots v_{i_s} = v_{j_1} \dots v_{j_t} \tag{4}$$

is a consequence (*in the class of all semigroups*) *of the relation* (2).

In other words, identity (3) holds in the variety $\mathfrak{K}$ defined by the axioms (1) if and only if the relation (4) holds in the semigroup $\mathfrak{A}$ presented by the generators v_i and defining relations (2).

Suppose that the relation (4) follows from the collection (2). Take any algebra $\mathfrak{C} = \langle C, \Omega \rangle$ in $\mathfrak{K}$ defined by the identities (1). The collection of all mappings $f: C \to C$ forms a semigroup $\mathfrak{C}_C$ (Section 3.1) relative to the composition operation $*$. Among the elements of $\mathfrak{C}_C$ are also the functions f_i ($i \in I$) of the algebra $\mathfrak{C}$. These functions satisfy the identities (1) which, with the help of the operation $*$, can be rewritten in the form

$$f_{i_{\lambda 1}} * \dots * f_{i_{\lambda s_\lambda}} = f_{j_{\lambda 1}} * \dots * f_{j_{\lambda t_\lambda}}. \tag{5}$$

By hypothesis, in every semigroup which contains elements v_i satisfying the relations (2), these elements also satisfy the relation (4). According to (5), the elements f_i satisfy precisely the relation (2) in the semigroup $\mathfrak{C}_c$. Therefore, the relation (4) must hold in $\mathfrak{C}_c$; consequently, the identity (3) holds in the class $\mathfrak{K}$.

Conversely, suppose that the identity (3) is a consequence of the identities (1). Let $\mathfrak{A}$ be any semigroup which contains elements v_i ($i \in I$) satisfying (2). Adjoining a new element e to $\mathfrak{A}$ satisfying the conditions $ee = e$ and $ex = xe = x$ ($x \in \mathfrak{A}$), we obtain a semigroup $\mathfrak{A}_e$ with identity element e which contains $\mathfrak{A}$ as a subsemigroup. With each element v_i, we associate a function $f_i(x)$ defined by

$$f_i(x) = v_i x \quad (x \in \mathfrak{A}_e). \tag{6}$$

From the relations (2), it is now seen that the functions f_i are connected in $\mathfrak{A}_e$ by the identities (1). By hypothesis, the identities (1) imply the identity (3). Setting $x = e$ in the latter and using equation (6), we obtain (4).

Theorem 1 shows that the mapping constructed above from varieties of $\mathfrak{m}$-unoids of the first kind onto the collection of all semigroups with $\mathfrak{m}$ distinguished generators is one-to-one. Here we deem two semigroups to be the same if there exists an isomorphism which takes the distinguished generating elements of one semigroup into like distinguished generators in the other semigroup.

Note that *under the above mapping, finitely axiomatizable varieties correspond to finitely presented semigroups. Also, if the variety $\mathfrak{R}$ contains a subvariety $\mathfrak{L}$, then the semigroup corresponding to the variety $\mathfrak{L}$ is a quotient semigroup of the semigroup which corresponds to $\mathfrak{R}$.*

The condition that the above semigroups must have $\mathfrak{m}$ distinguished generators in their signature is essential. It is clear that distinct varieties may correspond to isomorphic semigroups (in the abstract sense) which differ only in the choice of generators.

We now wish to make use of the correspondence between varieties of unoids and semigroups to determine the number of distinct varieties of algebras of a given signature.

Theorem 2. *The collection of varieties of 1-unoids is countable. The collection of all varieties of algebras of fixed signature Ω which contains more than one operator symbol or one non-unary operator symbol, has cardinality $2^{|\Omega|+\aleph_0}$.*

First, consider 1-unoids. Their signature consists of one unary operator symbol f. We introduce the notation

$$f^0x = x, \qquad f^kx = f \dots f(x).$$

Let $\mathfrak{M}$ be some variety of 1-unoids. Among all identities of the form

$$f^kx = f^lx \qquad (k < l;\ k, l = 0, 1, 2, \dots), \tag{7}$$

which hold in $\mathfrak{M}$, we look for those for which the number l is least and of these, we pick the one for which k is least. Suppose that the identity thus selected is $f^sx = f^tx$. Reasoning just as in Section 3.1 in the study of semigroups with one generator, we easily see that all the identities of the form (7) which hold in $\mathfrak{M}$ are formal consequences of the minimal identity $f^sx = f^tx$.

Analogously, if some identity of the form

$$f^mx = f^ny \qquad (m \leqq n;\ m, n = 0, 1, 2, \dots) \tag{8}$$

holds in $\mathfrak{M}$, then we quickly note that it is formally equivalent to the identity

$$f^mx = f^my. \tag{9}$$

Among all identities of the form (9) which hold in $\mathfrak{M}$, we choose the one in which m is least. Suppose that (9) is that identity. Then (9) will formally imply not only all identities of the form (8) which hold in $\mathfrak{M}$, but, also all identities of the form (7). Indeed, (9) implies the identity $f^m x = f^{m+1} x$. Therefore, in the minimal identity $f^s x = f^t x$ $(s < t)$, we have $t \leqq m + 1$ and $s \leqq m$. From $f^s x = f^t x$ and (9) follows $f^s x = f^r x = f^m y$ $\left(r = s + (m + 1)(t - s)\right)$ and the latter implies the identity $f^s x = f^s y$. Therefore, $s = m$ and the identity $f^s x = f^t x$ is a formal consequence of the identity $f^m x = f^m y$.

Thus every variety of 1-unoids is defined either by one identity of the form (7) or one identity of the form (9). The set of identities of these forms is countable and distinct identities define distinct varieties.

Let us consider the case $|\Omega| \geqq 2$. The number of distinct varieties of signature Ω certainly does not exceed the number of axiomatizable classes of this signature, i.e. is not more than $2^{|\Omega|+\aleph_0}$ (see 1, Problems and Complements, Section 6). Therefore, to prove the assertion, we must construct $2^{|\Omega|+\aleph_0}$ distinct varieties of signature Ω.

First, suppose that Ω consists only of one-place operator symbols f_i $(i \in I)$. Above, we constructed a one-to-one correspondence between varieties of Ω-algebras of the first kind and semigroups with distinguished sets of generators v_i $(i \in I)$. The collection of all nonisomorphic semigroups with $|\Omega| \geqq 2$ generators has cardinality $2^{|\Omega|+\aleph_0}$ (see 3, Problems and Complements, Section 3). Therefore, the collection of all varieties of Ω-unoids also has cardinality $2^{|\Omega|+\aleph_0}$.

Now suppose $|\Omega| \geqq 2$ and Ω contains a certain number of nonunary operator symbols f_j $(j \in J)$. Let n_j denote the arity of f_j. Consider the subvarieties of the variety defined by the identities

$$f_j(x_1, x_2, \ldots, x_{n_j}) = f_j(x_1, y_2, \ldots, \ldots, y_{n_j}) \qquad (j \in J), \tag{10}$$

which say that in the variety (10) all basic operations are essentially one-place functions. With each symbol $f_i \in \Omega$, we associate a new one-place operator symbol g_i and consider unoids of signature $\Omega^* = \langle g_i \mid i \in I \rangle$. Each Ω^*-unoid can be made into an algebra belonging to the variety (10) by defining

$$f_i(x_1, \ldots, x_{n_i}) = g_i(x_1) \qquad (i \in I).$$

It is clear that this converts varieties of Ω^*-unoids into varieties of Ω-algebras which are in the variety (10). We have already seen that for $|\Omega^*| \geqq 2$, the collection of the varieties of Ω^*-unoids has cardinality $2^{|\Omega^*|+\aleph_0}$. Therefore, also the collection of varieties of Ω-algebras has the same cardinality for $|\Omega| \geqq 2$.

It remains to consider the case where the signature consists of one multi-place operator symbol. First we suppose that this symbol is

binary, i.e. that we are studying groupoids. Suppose that the relations (1) define the variety $\mathfrak{K}$ of 2-unoids of signature $\Omega = \langle f_1, f_2 \rangle$. In each of these relations, we replace f_1 and f_2 by the following:

$$f_1(x) = xx, \qquad f_2(x) = (xx)x. \tag{11}$$

This yields a system of identities with the signature of the groupoid. It determines a certain variety of groupoids $\mathfrak{K}_g$. If we have some $\mathfrak{K}_g$-groupoid $\langle A, \cdot \rangle$, then defining the operations f_1 and f_2 on A through formulas (11), we obtain a 2-unoid $\langle A, f_1, f_2 \rangle$ which belongs to the variety $\mathfrak{K}$. However, this unoid will satisfy the quasi-identity

$$f_1(x) = x \to f_2(x) = x. \tag{12}$$

Conversely, suppose $\mathfrak{K}^*$ denotes the quasivariety of $\mathfrak{K}$-unoids which satisfy requirement (12) and suppose $\mathfrak{A} = \langle A, f_1, f_2 \rangle$ is some $\mathfrak{K}^*$-unoid. Then defining a multiplication on the set A by the formulas

$$xx = f_1(x), \qquad xy = f_2(y) \qquad (x \neq y),$$

we obtain a $\mathfrak{K}_g$-groupoid from which the 2-unoid $\mathfrak{A} = \langle A, f_1, f_2 \rangle$ can be recovered through formulas (11). This shows that if for varieties $\mathfrak{K}_1$ and $\mathfrak{K}_2$ of 2-unoids the corresponding quasivarieties $\mathfrak{K}_1^*$ and $\mathfrak{K}_2^*$ are distinct, then the varieties $\mathfrak{K}_{1g}$ and $\mathfrak{K}_{2g}$ of groupoids will also be distinct.

We take an arbitrary semigroup $\mathfrak{G}$ with a countable number of generators $v_1, v_2, \ldots$ and a collection of defining relations of the form (2). In these relations, we substitute the expressions

$$v_i = a_1 a_2 a_1^{i+1} a_2^{i+1} \qquad (i = 1, 2, \ldots), \tag{13}$$

and obtain a system of defining relations for some semigroup $\mathfrak{G}$ with generators a_1 and a_2. We have seen (see 2, Problems and Complements, Section 11) that the formulas (13) define an embedding of semigroup $\mathfrak{G}$ into semigroup $\mathfrak{G}_0$.

Suppose $\mathfrak{G}$ and $\mathfrak{H}$ are *distinct* semigroups with generators $v_1, v_2, \ldots$ (distinct in the sense that there is no isomorphism $\varphi: \mathfrak{G} \to \mathfrak{H}$ under which $v_i\varphi = v_i$). Then $\mathfrak{G}_0$ and $\mathfrak{H}_0$ will be distinct semigroups with generators a_1 and a_2. By Theorem 1, to the semigroups $\mathfrak{G}_0$ and $\mathfrak{H}_0$, there correspond the varieties $\mathfrak{K}$ and $\mathfrak{L}$ of 2-unoids of signature $\langle f_1, f_2 \rangle$. We pick out of $\mathfrak{K}$ and $\mathfrak{L}$, the quasivarieties $\mathfrak{K}^*$ and $\mathfrak{L}^*$ of those unoids in which the quasi-identity (12) holds. Let us show that $\mathfrak{K}^* \neq \mathfrak{L}^*$.

We adjoin identity elements to the semigroups $\mathfrak{G}_0$ and $\mathfrak{H}_0$ and we construct 2-unoids $\mathfrak{A} \in \mathfrak{K}$ and $\mathfrak{B} \in \mathfrak{L}$ by defining the operations

$$f_1(x) = a_1 x, \qquad f_2(x) = a_2 x$$

on the carriers of $\mathfrak{G}_0$ and $\mathfrak{H}_0$.

Since the inequality $a_1 x \neq x$ is true for any x both in the semigroup $\mathfrak{G}_0$ and in the semigroup $\mathfrak{H}_0$, the quasi-identity (12) holds in the unoids $\mathfrak{A}$ and $\mathfrak{B}$, i.e. $\mathfrak{A} \in \mathfrak{K}^*$ and $\mathfrak{B} \in \mathfrak{L}^*$. But $\mathfrak{A} \notin \mathfrak{L}$ and $\mathfrak{B} \notin \mathfrak{K}$ and hence $\mathfrak{K}^* \neq \mathfrak{L}^*$.

We know that there is a continuum of distinct semigroups $\mathfrak{G}_i$ with a countable set of signature generators. Their corresponding semigroups $\mathfrak{G}_{i0}$ with two signature generators are also distinct. In turn, the varieties of groupoids corresponding to them are also distinct. From this, we conclude that the collection of varieties of groupoids has the cardinality of the continuum.

Finally, if the signature of the algebras under consideration consists of one n-ary operator symbol f and $n > 2$, then we take the variety $\mathfrak{M}$ of algebras which satisfy the identity

$$f(x_1, x_2, x_3, \ldots, x_n) = f(x_1, x_2, y_3, \ldots, y_n).$$

Every semigroup can be made into an $\mathfrak{M}$-algebra by setting

$$f(x_1, x_2, \ldots, x_n) = x_1 x_2.$$

Distinct varieties of semigroups then become distinct varieties of $\mathfrak{M}$-algebras. Therefore, the collection of varieties with given signature $\langle f \rangle$ has the cardinality of the continuum.

Theorem 2 deals with varieties of *algebras*. We shall make several remarks about varieties of *systems*. First of all, if the signature Ω is infinite or the set of its operator symbols is nonempty and has more than just a single one-place symbol, then the collection of all varieties of signature Ω, obviously, will also have cardinality $2^{|\Omega|+\aleph_0}$.

If Ω consists of a finite number of predicate symbols P_i, then the identities will have the form

$$P_i(x_\alpha, \ldots, x_\gamma) = \mathsf{T},$$

they will be finite in number and so the number of distinct varieties will also be a finite number.

Suppose the signature Ω consists of a one-place function f and one-place predicates P_i $(i = 1, \ldots, n)$. It was shown above that any collection of identities with signature $\langle f \rangle$ is equivalent to a single identity of the form (7) or (9). The identities containing the symbol P_i have the form

$$P_i(f^s x) = \mathsf{T}$$

and the entire collection of them is clearly equivalent to the set of those in which s is least. Thus, the variety of signature Ω is determined by at most $n + 1$ identities, and hence the collection of these varieties is at most countable.

We turn, finally, to the case where the signature Ω consists of a one-place function f and predicates P_i. Among the latter there are ones which are not one-place predicates. Suppose, for example, that Ω contains an n-ary predicate P ($n \geqq 2$). Let D be a subset of the positive integers. Consider the variety $\mathfrak{K}_D$ defined by the identities

$$P(x, f^d x, \ldots, f^d x) = \mathsf{T} \qquad (d \in D).$$

Distinct subsets D and D_1 correspond to distinct varieties $\mathfrak{K}_D$ and $\mathfrak{K}_{D_1}$. Indeed, let

$$\mathfrak{A}_D = \langle \{0, 1, 2, \ldots\}; f, P, \ldots \rangle,$$

where $f(x) = x + 1$ and

$$P(x, y, \ldots, z) \Leftrightarrow y - x = \ldots = z - x \in D,$$

$$P_i(x, y, \ldots, z) = \mathsf{T} \qquad (P_i \neq P,\ P_i \in \Omega).$$

It is clear that $\mathfrak{A}_D \neq \mathfrak{A}_{D_1}$ for $D \neq D_1$ and $\mathfrak{A}_D \in \mathfrak{K}_{D_1} \Leftrightarrow D \supseteq D_1$. Thus

$$\mathfrak{K}_D = \mathfrak{K}_{D_1} \Leftrightarrow D = D_1.$$

The number of distinct subsets of the positive integers is equal to $2^{\aleph_0}$. Consequently, there are that many distinct varieties of signature Ω.

Problems and Complements

1. The class $H\mathfrak{K}$ of all homomorphic images of systems of any replete class $\mathfrak{K}$ is a variety.

2. Let $\mathfrak{A}_2$ be the variety of groups defined (in the class of all groups) by the identity $x^2 = 1$. The class $\mathfrak{A}_2^2$ of groups $\mathfrak{G}$ which have a normal subgroup N such that N and $\mathfrak{G}/N$ belong to $\mathfrak{A}_2$, is a variety with infinite basis rank (B. Neumann, H. Neumann, P. Neumann [54])*).

14. Primitive Closures

The *primitive closure* of a class of systems $\mathfrak{K}$ is the minimal variety $\hat{\mathfrak{K}} = KI\mathfrak{K}$ containing all systems of the class $\mathfrak{K}$. The class $\mathfrak{K}$ is called the *generating class* of the variety $\hat{\mathfrak{K}}$. The preceding sections have already established the existence of the primitive closure of any class $\mathfrak{K}$, as well as the existence of the analogously defined *axiomatic, quasiprimitive* and *replete closures* of $\mathfrak{K}$. We now wish to study in greater detail how various properties of the class $\mathfrak{K}$ affect the variety $\hat{\mathfrak{K}}$ which it generates.

*) The first example of a group variety of infinite basis rank was given by G. Higman [20].

We turn, finally, to the case where the signature Ω consists of a one-place function f and predicates P_i. Among the latter there are ones which are not one-place predicates. Suppose, for example, that Ω contains an n-ary predicate P ($n \geqq 2$). Let D be a subset of the positive integers. Consider the variety $\mathfrak{K}_D$ defined by the identities

$$P(x, f^d x, \ldots, f^d x) = \mathsf{T} \qquad (d \in D).$$

Distinct subsets D and D_1 correspond to distinct varieties $\mathfrak{K}_D$ and $\mathfrak{K}_{D_1}$. Indeed, let

$$\mathfrak{A}_D = \langle \{0, 1, 2, \ldots\}; f, P, \ldots \rangle,$$

where $f(x) = x + 1$ and

$$P(x, y, \ldots, z) \Leftrightarrow y - x = \ldots = z - x \in D,$$

$$P_i(x, y, \ldots, z) = \mathsf{T} \qquad (P_i \neq P,\ P_i \in \Omega).$$

It is clear that $\mathfrak{A}_D \neq \mathfrak{A}_{D_1}$ for $D \neq D_1$ and $\mathfrak{A}_D \in \mathfrak{K}_{D_1} \Leftrightarrow D \supseteq D_1$. Thus

$$\mathfrak{K}_D = \mathfrak{K}_{D_1} \Leftrightarrow D = D_1.$$

The number of distinct subsets of the positive integers is equal to $2^{\aleph_0}$. Consequently, there are that many distinct varieties of signature Ω.

Problems and Complements

1. The class $H\mathfrak{K}$ of all homomorphic images of systems of any replete class $\mathfrak{K}$ is a variety.

2. Let $\mathfrak{A}_2$ be the variety of groups defined (in the class of all groups) by the identity $x^2 = 1$. The class $\mathfrak{A}_2^2$ of groups $\mathfrak{G}$ which have a normal subgroup N such that N and $\mathfrak{G}/N$ belong to $\mathfrak{A}_2$, is a variety with infinite basis rank (B. Neumann, H. Neumann, P. Neumann [54])*).

14. Primitive Closures

The *primitive closure* of a class of systems $\mathfrak{K}$ is the minimal variety $\hat{\mathfrak{K}} = KI\mathfrak{K}$ containing all systems of the class $\mathfrak{K}$. The class $\mathfrak{K}$ is called the *generating class* of the variety $\hat{\mathfrak{K}}$. The preceding sections have already established the existence of the primitive closure of any class $\mathfrak{K}$, as well as the existence of the analogously defined *axiomatic, quasiprimitive* and *replete closures* of $\mathfrak{K}$. We now wish to study in greater detail how various properties of the class $\mathfrak{K}$ affect the variety $\hat{\mathfrak{K}}$ which it generates.

*) The first example of a group variety of infinite basis rank was given by G. Higman [20].

14.1 Generating Systems. Let $\mathfrak{K}$ be a variety and let $\mathfrak{F}_\mathfrak{m}$ be a free system of rank $\mathfrak{m}$ which belongs to $\mathfrak{K}$. According to Section 13.1,

$$\mathfrak{K} = KI\mathfrak{F}_\omega = \hat{\mathfrak{F}}_\omega,$$

i.e. *every variety has a generating system* which, in particular, is a free system of countable rank.

Is every variety generated by some collection of free systems of finite rank?

Theorem 1. *Every variety $\mathfrak{K}$ is generated by the collection of all its free systems of finite rank. If it is generated by some collection of systems whose basis ranks do not exceed a finite number n, then $\mathfrak{K}$ is generated by free system $\mathfrak{F}_n$.*

Suppose that the elements $v_1, v_2, \ldots$ are a free basis for $\mathfrak{F}_\omega$. Denoting by $\mathfrak{F}_n$ the subsystem generated in $\mathfrak{F}_\omega$ by the elements $v_1, v_2, \ldots, v_n$, we have $\mathfrak{F}_\omega = \bigcup \mathfrak{F}_n$, where $\mathfrak{F}_n$ is a free system of rank n. From this, it is clear that an arbitrary identity holds in $\mathfrak{F}_\omega$ if and only if it holds in each of the subsystems $\mathfrak{F}_n$ $(n = 1, 2, \ldots)$. This proves the first assertion of Theorem 1. The second assertion follows from the remark that every system of finite basis rank n is a homomorphic image of the system $\mathfrak{F}_n$. Therefore,

$$I\mathfrak{F}_n \subseteq I\{\mathfrak{A}_j \mid j \in J\} \subseteq I\mathfrak{F}_n,$$

where the $\mathfrak{A}_j$ $(j \in J)$ are the given systems. Hence,

$$KI\mathfrak{F}_n = KI\{\mathfrak{A}_j \mid j \in J\}.$$

Varieties which can not be generated by its free systems of finite rank were called varieties with infinite basis rank in Section 13.2. An example of such a variety was given in Problem 2, Section 13.

We know that a class of systems need not be a set with well-defined cardinality, even if the systems are viewed up to isomorphism. Suppose that the class of systems $\mathfrak{K} = \langle \mathfrak{A}_i \mid i \in I \rangle$ is a set. According to Section 13.1, the free systems of the variety generated by the set $\mathfrak{K}$ are at the same time free systems of the replete class $S\prod \mathfrak{K}_e$ generated by $\mathfrak{K}$. All systems of the replete class, including all free systems, are subsystems of direct products of systems of the set $\mathfrak{K}_e$. How many factors need we take in order that a product would be certain to contain a free system of given rank $\mathfrak{m}$? The answer is given by

Theorem 2. *Suppose that $\mathfrak{F}_\mathfrak{m}$ is a free system of rank $\mathfrak{m}$ in the variety $\hat{\mathfrak{K}}$ generated by the set of systems $\mathfrak{K} = \{\mathfrak{A}_i \mid i \in I\}$. Then, up to isomor-*

phism,

$$\mathfrak{F}_{\mathfrak{m}} \subseteq \prod \mathfrak{A}_i^{|\mathfrak{A}_i|^{\mathfrak{m}}} \qquad (i \in I). \tag{1}$$

Let M be any set of cardinality $\mathfrak{m}$. Denote by L_i the collection of all possible mappings $\gamma: M \to \mathfrak{A}_i$, i.e. $L_i = \mathfrak{A}_i^M$. The cardinality of L_i satisfies

$$|L_i| = |\mathfrak{A}_i|^{\mathfrak{m}}.$$

With each $\gamma \in L_i$, we associate the same system $\mathfrak{A}_i$ but the system corresponding to γ will be viewed as a copy of $\mathfrak{A}_i$ numbered γ, denoted by the symbol $\mathfrak{A}_{i\gamma}$. Thus, we have a collection of mappings

$$\gamma: M \to \mathfrak{A}_{i\gamma} \qquad (\gamma \in L_i). \tag{2}$$

These mappings uniquely determine the mapping

$$\beta_i: M \to \prod_{\gamma \in L_i} \mathfrak{A}_{i\gamma} = \mathfrak{A}_i^{|L_i|}.$$

In turn, the collection of mappings $\beta (i \in I)$ uniquely determines a mapping

$$\alpha: M \to \prod_i \prod_\gamma \mathfrak{A}_{i\gamma} = \prod \mathfrak{A}_i^{|L_i|}$$

satisfying

$$\gamma = \alpha \pi_{i\gamma} \qquad (\gamma \in L_i,\ i \in I), \tag{3}$$

where the $\pi_{i\gamma}$ are projections of the system $\mathfrak{A} = \prod_{i,\gamma} \mathfrak{A}_{i\gamma}$ onto the factors $\mathfrak{A}_{i\gamma}$.

Denote by $\mathfrak{B}$ the subsystem of $\mathfrak{A}$ generated by the elements $b_m = m\alpha$ $(m \in M)$. We must prove now that $\mathfrak{B}$ is a system of rank $\mathfrak{m}$ free in $\hat{\mathfrak{K}}$. To this end, we must show that the collection of elements $b_m (m \in M)$ is $\mathfrak{K}$-independent in $\mathfrak{A}$ since independence relative to $\mathfrak{K}$ implies, by Theorem 3, Section 12.2, independence with respect to $HS\prod \mathfrak{K} = \hat{\mathfrak{K}}$.

Consider an arbitrary mapping $\varphi: b_m \to \mathfrak{A}_i$ $(m \in M)$. The corresponding mapping $m \to b_m\varphi \in \mathfrak{A}_i$ must coincide with some mapping (2) and hence

$$b_m\varphi = m\gamma \qquad (m \in M).$$

By (3) from this we obtain

$$b_m\varphi = m\gamma = m\alpha\pi_{i\gamma} = b_m\pi_{i\gamma},$$

i.e. the mapping φ can be extended to a homomorphism $\pi_{i\gamma}: \mathfrak{A} \to \mathfrak{A}_i$ and thus to a homomorphism $\pi_{i\gamma}: \mathfrak{B} \to \mathfrak{A}_i$, as required.

From Theorem 2, it follows that *if a variety is generated by a single system* $\mathfrak{A}$, *then for a system* $\mathfrak{F}_{\mathfrak{m}}$ *of rank* $\mathfrak{m}$ *which is free in this variety, we*

have the formula

$$\mathfrak{F}_m \subseteq \mathfrak{A}^{|\mathfrak{A}|^m}. \tag{4}$$

Let us agree to say that a family of systems is *locally finite* if each finite set of elements of an arbitrary system in this collection generates a finite subsystem. A family of systems will be called *uniformly locally finite* if for each finite m there exists a finite $n = \varphi(m)$ such that any arbitrary m elements of any system from the given family generate a subsystem whose cardinality does not exceed n.

Theorem 3. *A variety $\mathfrak{M}$ of finite signature is locally finite if and only if it is generated by a uniformly locally finite family of systems. If a variety is locally finite, then it is uniformly locally finite.*

Necessity. Suppose that the variety $\mathfrak{M}$ is locally finite. Then the cardinality of its free system $\mathfrak{F}_m$ of finite rank m will be a finite number $\varphi(m)$. Any $\mathfrak{M}$-system generated by m elements will be a homomorphic image of the free system $\mathfrak{F}_m$ and hence will have cardinality not greater than $\varphi(m) = |\mathfrak{F}_m|$. This proves the necessity of the condition and the last assertion in Theorem 3.

Sufficiency. Suppose that the variety $\mathfrak{M}$ is generated by a uniformly locally finite class $\mathfrak{K}$. Then for a suitable set I and mapping of $I \to \mathfrak{K}$, we have

$$\mathfrak{F}_m \subseteq \prod \mathfrak{A}_i \quad (i \in I,\ \mathfrak{A}_i \in \mathfrak{K}).$$

Suppose that m is finite and let $v_1, \ldots, v_m$ be a free basis for the system $\mathfrak{F}_m$ and π_i the projection of the product $\prod \mathfrak{A}_i$ onto $\mathfrak{A}_i$. Denote by $\mathfrak{B}_i$ the subsystem generated in $\mathfrak{A}_i$ by the elements $v_1\pi_i, \ldots, v_m\pi_i$. Then

$$\mathfrak{F}_m \subseteq \prod \mathfrak{B}_i \quad (i \in I,\ \mathfrak{B}_i \in \mathfrak{M}).$$

From this, it follows that $\mathfrak{F}_m$ is a free system in the variety $\mathfrak{L}$ generated by the systems $\mathfrak{B}_i$ $(i \in I)$. All these systems have finite signature and the cardinality of each does not exceed a certain finite number $\varphi(m)$. This implies that among these systems there is only a finite number of mutually nonisomorphic ones. Let these be the systems $\mathfrak{B}_1, \ldots, \mathfrak{B}_t$. We see that the variety $\mathfrak{L}$ is generated by the systems $\mathfrak{B}_1, \ldots, \mathfrak{B}_t$ and that $\mathfrak{F}_m$ is a free system in $\mathfrak{L}$. According to formula (1)

$$\mathfrak{F}_m \subseteq \mathfrak{B}_1^{|\mathfrak{B}_1|^m} \times \ldots \times \mathfrak{B}_t^{|\mathfrak{B}_t|^m},$$

i.e. the system $\mathfrak{F}_m$ is certainly finite.

Every $\mathfrak{M}$-system generated by m elements is a homomorphic image of $\mathfrak{F}_m$, and hence the cardinality of each such system does not exceed the finite number $|\mathfrak{F}_m|$.

The condition in Theorem 3 that the signature be finite is essential. To see this, consider the collection of algebras

$$\mathfrak{A}_n = \langle \{0, 1\}; f_1, f_2, \ldots \rangle,$$

where $f_1, f_2, \ldots$ are one-place functions such that in $\mathfrak{A}_n$

$$0f_i = 0, \quad 0f_{n+j} = 1 \qquad (i < n;\ j = 0, 1, \ldots).$$

Clearly, the collection of algebras $\mathfrak{A}_n$ is uniformly locally finite. Denote by $\mathfrak{M}$ the variety generated by this collection and let v be the free generating element for the $\mathfrak{M}$-free algebra $\mathfrak{F}_1$. Let us show that in $\mathfrak{F}_1$, the elements $vf_1, vf_2, \ldots$ are pairwise distinct. Indeed, if it were true that $vf_i = vf_n$ $(i < n)$, then the identity $xf_i = xf_n$ would hold in all algebras $\mathfrak{A}_j$, which is certainly false for $x = 0$ in the algebra $\mathfrak{A}_n$. Thus, all the free algebras in the variety $\mathfrak{M}$ are infinite, although this variety is generated by a locally finite collection of algebras.

On the other hand, the finiteness requirement on the signature may be relaxed if the variety is generated by a single finite system:

Theorem 4. *A finite system of any signature generates a locally finite variety.*

Suppose the variety $\mathfrak{M}$ is generated by the finite system $\mathfrak{A}$. From formula (4), we see that all the free systems in $\mathfrak{M}$ of finite rank are finite and hence the variety $\mathfrak{M}$ is locally finite.

In particular, Theorem 4 implies that *the direct powers of any finite algebraic system are locally finite.*

Recall that we say a system $\mathfrak{A}$ can be finitely approximated, if for each $P \in \{\Omega, =\}$ and each n-tuple of elements $a_1, \ldots, a_n$ for which $P(a_1, \ldots, a_n) = \mathsf{F}$, there exists a homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{M}$ onto a finite system $\mathfrak{M}$ such that $P(a_1\varphi, \ldots, a_n\varphi) = \mathsf{F}$.

Varieties generated by arbitrary collections of finite systems have a number of interesting properties. A basic property is given by

Theorem 5. *A variety $\mathfrak{M}$ can be generated by finite systems if and only if all its free systems can be finitely approximated.*

Suppose $\mathfrak{M}$ is generated by a collection of finite systems $\mathfrak{A}_i (i \in I)$. Then by formula (1), every free system in $\mathfrak{M}$ will be a subsystem of a direct product of the $\mathfrak{A}_i$. Hence, by Section 2.5, it can be finitely approximated.

Conversely, suppose that some $\mathfrak{M}$-free system $\mathfrak{F}_m$ can be finitely approximated. By Section 2.5, this means that

$$\mathfrak{F}_m \subseteq \prod \mathfrak{C}_i \qquad (i \in I),$$

where the $\mathfrak{C}_i$ are finite homomorphic images of $\mathfrak{F}_m$. From $\mathfrak{F}_m \in \mathfrak{M}$ and the homomorphic closure of $\mathfrak{M}$, it follows that $\mathfrak{C}_i \in \mathfrak{M}$. Now let $\mathfrak{N}$ denote the variety generated by all finite systems belonging to $\mathfrak{M}$; we see that $\mathfrak{F}_m \in \mathfrak{N}$. Since $\mathfrak{M}$ and $\mathfrak{N}$ have the same free systems, $\mathfrak{M} = \mathfrak{N}$.

Lemma 6. *If a system $\mathfrak{A}$ can be finitely approximated and is generated by a finite set of its elements, then every homomorphism σ of this system onto itself is an isomorphism.*

Suppose that for some $P \in \langle \Omega, = \rangle$ and $a_1, \ldots, a_n \in \mathfrak{A}$, we have $P(a_1, \ldots, a_n) = \mathsf{F}$. We must show that then $P(a_1\sigma, \ldots, a_n\sigma) = \mathsf{F}$.

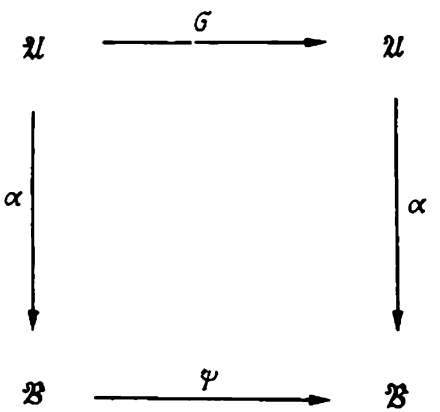

Fig. 1

Consider the homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{M}$, where $\mathfrak{M}$ is a finite system and $P(a_1\varphi, \ldots, a_n\varphi) = \mathsf{F}$. The replete class $\mathfrak{K} = S\prod(\mathfrak{M})_s$ consists of the subsystems of direct powers of finite systems. But direct powers of a finite system are locally finite. Therefore, all systems of the class $\mathfrak{K}$ are locally finite. Let $\mathfrak{B}$ denote a replica of the system $\mathfrak{A}$ in $\mathfrak{K}$ and suppose $\alpha\colon \mathfrak{A} \to \mathfrak{B}$ is the corresponding $\mathfrak{K}$-morphism. According to Theorem 1, there exists an epimorphism $\psi\colon \mathfrak{B} \to \mathfrak{B}$ such that the diagram in Fig. 1 is commutative. The system $\mathfrak{B}$ is a homomorphic image of the finitely generated system $\mathfrak{A}$ and hence $\mathfrak{B}$ itself is finitely generated. But $\mathfrak{B}$ belongs to the class $\mathfrak{K}$ which consists only of locally finite systems. Consequently, the system $\mathfrak{B}$ is finite. By Section 2.2, every homomorphism of a finite system onto itself is an isomorphism, and we conclude that the homomorphism ψ is an isomorphism. Now consider the homomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{M}$. Since $\mathfrak{M} \in \mathfrak{K}$, there is a homomorphism $\xi\colon \mathfrak{B} \to \mathfrak{M}$ satisfying $\varphi = \alpha\xi$. By hypothesis, $P(a_1\varphi, \ldots, a_n\varphi) = P(a_1\alpha\xi, \ldots, a_n\alpha\xi) = \mathsf{F}$. Therefore, $P(a_1\alpha, \ldots, a_n\alpha) = \mathsf{F}$. Since ψ is an isomorphism,

we have $P(a_1\alpha\psi, \ldots, a_n\alpha\psi) = \mathsf{F}$, i.e. $P(a_1\sigma\alpha, \ldots, a_n\sigma\alpha) = \mathsf{F}$ and hence $P(a_1\sigma, \ldots, a_n\sigma) = \mathsf{F}$.

Lemma 6 and Theorem 4 of Section 13.2 lead to the next result.

Corollary 7. *If a system $\mathfrak{F}_n$ of finite rank n is free in itself and can be finitely approximated, then any n elements of $\mathfrak{F}_n$ which generate $\mathfrak{F}_n$ form a free basis for $\mathfrak{F}_n$.*

Combining this assertion with Theorem 5, we obtain

Corollary 8 (Jónsson-Tarski [23]). *If a variety $\mathfrak{M}$ is generated by its finite systems, then in an $\mathfrak{M}$-free system $\mathfrak{F}_m$ of arbitrary finite rank m, any m elements which generate $\mathfrak{F}_m$ form a free basis of $\mathfrak{F}_m$.*

Theorems 2—5 and Corollary 8 are statements for varieties. However, from their proofs, it is immediate that the same assertions hold also for quasivarieties and arbitrary replete classes. Such is not the case with Theorem 1. Its proof is based on the fact that a free system of countable rank in a variety uniquely determines that variety; this is not true for quasivarieties.

Theorem 9. *A quasivariety $\mathfrak{K}$ has a generating system if and only if for any two nontrivial $\mathfrak{K}$-systems $\mathfrak{B}$ and $\mathfrak{C}$, there exists a $\mathfrak{K}$-system containing subsystems respectively isomorphic to $\mathfrak{B}$ and $\mathfrak{C}$.*

Sufficiency. Suppose that any two systems in the quasivariety $\mathfrak{K}$ are contained in a suitable third $\mathfrak{K}$-system. By Theorem 6, Section 12.1, this implies that for any collection of nontrivial $\mathfrak{K}$-systems $\mathfrak{A}_i (i \in I)$, there is a $\mathfrak{K}$-system containing subsystems which are respectively isomorphic to the given systems $\mathfrak{A}_i$.

As usual, let $Q\mathfrak{K}$ denote the collection of all quasi-identities which hold in $\mathfrak{K}$. Let R be the collection of all other quasi-identities of given signature which do not hold in $\mathfrak{K}$. For each formula $\mathscr{A} \in R$, select a system $\mathfrak{S}_{\mathscr{A}}$ in $\mathfrak{K}$ in which $\mathscr{A}$ is false. According to the remark just made, there will be a system $\mathfrak{A}$ in $\mathfrak{K}$ in which any of the systems $\mathfrak{S}_{\mathscr{A}}$ can be isomorphically embedded. Since the quasi-identity $\mathscr{A}$ is false in $\mathfrak{S}_{\mathscr{A}}$ and $\mathfrak{S}_{\mathscr{A}} \subseteq \mathfrak{A}$, we see that $\mathscr{A}$ is also false in $\mathfrak{A}$, i.e. $Q\mathfrak{A} \cap R = \varnothing$. $\mathfrak{A} \in \mathfrak{K}$ implies that $Q\mathfrak{A} \supseteq Q\mathfrak{K}$. Therefore, $Q\mathfrak{A} = Q\mathfrak{K}$, and this means that the quasivariety $\mathfrak{K}$ is generated by the system $\mathfrak{A}$.

Necessity. Suppose that a quasivariety $\mathfrak{K}$ is generated by $\mathfrak{A}$. By Theorem 4 of Section 11.1, $\mathfrak{K}$ consists of the trivial system and all possible subsystems of filtered powers of $\mathfrak{A}$. Choose arbitrary nontrivial $\mathfrak{K}$-systems $\mathfrak{B}$ and $\mathfrak{C}$ and suppose that

$$\mathfrak{B} \subseteq \prod \mathfrak{A}/D_1, \qquad \mathfrak{C} \subseteq \prod \mathfrak{A}/D_2,$$

where D_1 and D_2 are suitable filters. Consider the system

$$\mathfrak{S} = \prod (\prod \mathfrak{A}/D_1)/D_2.$$

It is clear that $\mathfrak{S} \in \mathfrak{K}$. Furthermore, the system $\prod \mathfrak{A}/D_1$ is canonically embedded in $\mathfrak{S}$. Hence the system $\mathfrak{B}$ is also embedded in $\mathfrak{S}$. On the other hand, the system $\mathfrak{A}$ is canonically embedded in $\prod \mathfrak{A}/D_1$. Therefore, $\mathfrak{S}$ contains the subsystem $\prod \mathfrak{A}/D_2$ and consequently, $\mathfrak{S}$ also contains the subsystem $\mathfrak{C}$.

Theorem 9 can be put into a different form. Indeed, by Section 12.1, each pair of nontrivial systems of a quasivariety $\mathfrak{K}$ is contained in a suitable $\mathfrak{K}$-system if and only if all $\mathfrak{K}$-compositions of nontrivial $\mathfrak{K}$-systems are injective. From this, we obtain

Corollary 10. *A quasivariety $\mathfrak{K}$ is generated by a single system if and only if the $\mathfrak{K}$-compositions of any nontrivial $\mathfrak{K}$-systems are injective.*

The stipulations concerning nontriviality made in Theorem 9 and Corollary 10 are essential. For example, the algebra

$$\mathfrak{A} = \langle \{1, 2, 3, \ldots\}, + \rangle$$

contains no trivial subsystems and hence the quasi-identity $x + x = x \to y = z$ holds in $\mathfrak{A}$, which shows that no nontrivial algebra belonging to the quasivariety $KQ\mathfrak{A}$ contains a trivial subalgebra.

In Section 12.1, we gave an example of a quasivariety $\mathfrak{K}$, in which there exist noninjective $\mathfrak{K}$-compositions of nontrivial $\mathfrak{K}$-systems. According to Corollary 10, $\mathfrak{K}$ is an example of a quasivariety which is not generated by any single system.

14.2 The Lattice of Varieties. Let us now view the collection of all varieties of signature Ω as a single entity. Denote this collection by M_Ω. Every variety is an axiomatizable class of systems. The collection of all axiomatizable classes of systems of signature Ω will be denoted by A_Ω.

The intersection of any collection of varieties is clearly a variety; also, among the varieties in M_Ω there is a largest—the total or absolute variety of all systems of signature Ω—and a smallest—the trivial variety, consisting only of the trivial system. Therefore, M_Ω is a complete lower semilattice with unit element and hence, M_Ω may be viewed as a complete lattice with a zero and unit element. Since the operation of intersection in A_Ω and M_Ω is the same, M_Ω is a lower subsemilattice of the lattice A_Ω. However, M_Ω is not a sublattice of the lattice A_Ω since the operations of lattice union of classes in A_Ω and in M_Ω are differently defined and in general are distinct from the set-theoretic union of classes. In what follows, we shall view M_Ω as a partially ordered set with set-theoretic inclusion of classes as the partial order relation.

In applications, one usually has to consider not M_Ω but the collection of all subvarieties of some fixed variety $\mathfrak{M}$. Such a collection will be denoted by $M(\mathfrak{M})$. We shall now examine its structure in greater detail.

A variety $\mathfrak{M}$ and each of its subvarieties $\mathfrak{K}$ have free systems of countable rank which we shall denote, respectively, by $\mathfrak{F}_{\mathfrak{M}} = \mathfrak{F}$ and $\mathfrak{F}_{\mathfrak{K}}$. By Section 13.1, the variety $\mathfrak{K}$ is uniquely determined by the system $\mathfrak{F}_{\mathfrak{K}}$. The question of which $\mathfrak{M}$-systems are free in a *specific* variety was also answered above: they are the systems free in themselves. Thus, there exists a one-to-one correspondence between $\mathfrak{M}$-systems free in themselves with a countable basis free in itself and subvarieties of the variety $\mathfrak{M}$. The correspondence is given by

$$\mathfrak{F}_{\mathfrak{K}} \to KI\mathfrak{F}_{\mathfrak{K}}.$$

The one to-one property is to be understood in the sense that isomorphic systems correspond to the same variety and nonisomorphic systems correspond to distinct varieties.

We now take a fixed specific free system $\mathfrak{F}$ of countable rank in the basic variety $\mathfrak{M}$. Let $\mathfrak{K}$ be any subvariety in $\mathfrak{M}$. According to Section 12.2, the corresponding $\mathfrak{K}$-free system $\mathfrak{F}_{\mathfrak{K}}$ is a $\mathfrak{K}$-replica of $\mathfrak{F}$ and hence there is a $\mathfrak{K}$-morphism

$$\varphi\colon \mathfrak{F} \to \mathfrak{F}_{\mathfrak{K}},$$

from which it follows that $\mathfrak{F}_{\mathfrak{K}}$ is isomorphic to a suitable quotient system $\mathfrak{F}/\theta$.*) The $\mathfrak{K}$-free systems $\mathfrak{F}_{\mathfrak{K}}$ are determined only up to isomorphism. However, their quotient systems $\mathfrak{F}/\theta$ will be the same.

Indeed, suppose $\mathfrak{F}_{\mathfrak{K}}$ and $\mathfrak{F}_{\mathfrak{K}^*}$ are free systems of countable rank in $\mathfrak{K}$ and φ and φ^* are the corresponding $\mathfrak{K}$-morphisms. We must show that the kernel congruences θ and θ^* defined by the $\mathfrak{K}$-morphisms φ and φ^* coincide. Consider the homomorphism $\xi\colon \mathfrak{F}_{\mathfrak{K}} \to \mathfrak{F}_{\mathfrak{K}^*}$ for which $\varphi^* = \varphi\xi$. If a and $b \in \mathfrak{F}$ and $a \equiv b(\theta)$, then $a\varphi = b\varphi$, from which $a\varphi\xi = b\varphi\xi$, i.e. $a\varphi^* = b\varphi^*$. Thus, $a \equiv b(\theta)$ implies $a \equiv b(\theta^*)$. The converse also holds and therefore $\theta = \theta^*$.

Thus, *the mapping*

$$\varkappa\colon \mathfrak{F}/\theta \to KI(\mathfrak{F}/\theta) \tag{1}$$

is a one-to-one correspondence between quotient systems $\mathfrak{F}/\theta$ of countable rank free in themselves and subvarieties of the basic variety $\mathfrak{M}$.

We shall now find conditions under which the quotient system $\mathfrak{F}/\theta$ is a system of countable rank, free in itself. Choose a fixed free basis $v_1, v_2, \ldots$

*) Indeed, let θ be the kernel congruence corresponding to the $\mathfrak{K}$-morphism φ and σ the canonical homomorphism of $\mathfrak{F}$ onto $\mathfrak{F}/\theta$. By definition of $\mathfrak{K}$-replica, there exists a homomorphism ξ of $\mathfrak{F}_{\mathfrak{K}}$ onto $\mathfrak{F}/\theta$ such that $\sigma = \varphi\xi$. Using the definition of quotient system $\mathfrak{F}/\theta$, one can easily verify that ξ is an isomorphism [Ed.].

in $\mathfrak{F}$. If $\mathfrak{F}/\theta$ belongs to some subvariety $\mathfrak{K}$, then the canonical homomorphism $\mathfrak{F} \to \mathfrak{F}/\theta$ will be a $\mathfrak{K}$-morphism. A $\mathfrak{K}$-morphism of a free system onto a free system takes a free basis into a free basis. Therefore, *a quotient system $\mathfrak{F}/\theta$ will be a free system of countable rank if and only if the sequence $v_1\theta, v_2\theta, \ldots$ forms a free basis in $\mathfrak{F}/\theta$.*

An arbitrary quotient system $\mathfrak{A}/\theta$ of a system $\mathfrak{A}$ is called *characteristic* if it remains invariant under any automorphism φ of $\mathfrak{A}$, i.e. if*)

$$P(a_1\theta_1, \ldots, a_n\theta) \Rightarrow P(a_1\varphi\theta, \ldots, a_n\varphi\theta) \tag{2}$$
$$(P \in \{\Omega, =\},\ a_1, \ldots, a_n \in \mathfrak{A}).$$

If condition (2) is fulfilled for any endomorphism φ of the system $\mathfrak{A}$ (i.e. homomorphism into itself), then the quotient system $\mathfrak{A}/\theta$ is called *fully characteristic.*

Theorem 1. *Let $\mathfrak{A}$ be a system of rank $\mathfrak{m}$ free in itself. Then the quotient system $\mathfrak{A}/\theta$ is also a system of rank $\mathfrak{m}$ free in itself if and only if $\mathfrak{A}/\theta$ is fully characteristic.*

Necessity. Let $v_i (i \in I)$ be a free basis for $\mathfrak{A}$ and φ some endomorphism of the system $\mathfrak{A}$; suppose that for $a_1, \ldots, a_n \in \mathfrak{A}$, we have

$$P(a_1\theta_1, \ldots, a_n\theta) = \mathsf{T} \quad (P \in \{\Omega, =\}). \tag{3}$$

Expressing the elements $a_1, \ldots, a_n$ in terms of the basis, we obtain

$$a_\lambda = f_\lambda(v_\alpha, \ldots, v_\gamma) \qquad (\lambda = 1, \ldots, n), \tag{4}$$

where $v_\alpha, \ldots, v_\gamma$ are distinct elements of the basis and $f_1, \ldots, f_n$ are suitable terms of signature Ω. Rewriting the relations (3) in the form

$$P\big(f_1(v_\alpha, \ldots, v_\gamma)\theta, \ldots, f_n(v_\alpha, \ldots, v_\gamma)\theta\big) = \mathsf{T} \tag{5}$$

and taking into account that the mapping $a \to a\theta$ is a homomorphism, we obtain

$$P\big(f_1(v_\alpha\theta, \ldots, v_\gamma\theta), \ldots, f_n(v_\alpha\theta, \ldots, v_\gamma\theta)\big) = \mathsf{T}. \tag{6}$$

By hypothesis, the collection $\{v_i\theta \mid i \in I\}$ is a free basis in $\mathfrak{A}/\theta$. Therefore, (6) implies that the identity

$$P\big(f_1(x_\alpha, \ldots, x_\gamma), \ldots, f_n(x_\alpha, \ldots, x_\gamma)\big) = \mathsf{T}$$

holds in $\mathfrak{A}/\theta$.

---

*) (2) is actually equivalent to the condition

$$a_1\theta a_2 \Rightarrow (a_1\varphi)\theta(a_2\varphi) \quad (a_1, a_2 \in \mathfrak{A})$$

[*Ed*].

Setting $x_j = v_j\varphi\theta$ $(j = \alpha, \ldots, \gamma)$ and making the substitution (4), we obtain the relation

$$P(a_1\varphi\theta, \ldots, a_n\varphi\theta) = \mathsf{T}$$

showing that the quotient system $\mathfrak{A}/\theta$ is fully characteristic.

Sufficiency. Suppose $\{v_i \mid i \in I\}$ is a free basis for the system $\mathfrak{A}$ and $\mathfrak{A}/\theta$ is a fully characteristic quotient system. We must prove that the collection of elements $v_i\theta$ $(i \in I)$ is free in $\mathfrak{A}/\theta$, i.e. that for arbitrary terms $f_1, \ldots, f_n$, the relation (6) implies the relation

$$P\big(f_1(a_\alpha\theta, \ldots, a_\gamma\theta), \ldots, f_n(a_\alpha\theta, \ldots, a_\gamma\theta)\big) = \mathsf{T}, \tag{7}$$

where $a_\alpha, \ldots, a_\gamma$ are arbitrary elements of $\mathfrak{A}$.

The elements v_i are free in $\mathfrak{A}$. Therefore the mapping

$$\varphi\colon v_i \to a_i \qquad (i = \alpha, \ldots, \gamma)$$

can be extended to an endomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{A}$. Since the mapping $a \to a\theta$ is a homomorphism, (6) implies (5). Since the quotient system $\mathfrak{A}/\theta$ is fully characteristic, from (5) with the help of (2) we obtain

$$P\big(f_1(v_\alpha, \ldots, v_\gamma)\varphi\theta, \ldots, f_n(v_\alpha, \ldots, v_\gamma)\varphi\theta\big) = \mathsf{T}$$

and consequently,

$$P\big(f_1(v_\alpha\varphi\theta, \ldots, v_\gamma\varphi\theta), \ldots, f_n(v_\alpha\varphi\theta, \ldots, v_\gamma\varphi\theta)\big) = \mathsf{T},$$

which coincides with (7).

Theorem 1 enables us to say that the mapping (1) is a one-to-one correspondence between the set of all fully characteristic quotient systems of the system $\mathfrak{F}$ and the set of all subvarieties of the variety $\mathfrak{M}$.

As already mentioned, the set $M(\mathfrak{M})$ of subvarieties is partially ordered by set-theoretic inclusion of classes. On the other hand, the collection of all quotient systems of the system $\mathfrak{F}$ is also partially ordered according to the rule: $\mathfrak{F}/\theta \leqq \mathfrak{F}/\sigma$ if and only if the identity mapping: $\varepsilon\colon \mathfrak{F} \to \mathfrak{F}$ induces a homomorphism $\hat{\varepsilon}\colon \mathfrak{F}/\theta \to \mathfrak{F}/\sigma$.*) Let us show that the mapping (1) reverses the partial order, i.e. that for any fully characteristic quotient systems $\mathfrak{F}/\theta$ and $\mathfrak{F}/\sigma$, we have

$$\mathfrak{F}/\theta \leqq \mathfrak{F}/\sigma \Leftrightarrow KI(\mathfrak{F}/\theta) \supseteq KI(\mathfrak{F}/\sigma). \tag{8}$$

Thus, suppose $\mathfrak{F}/\theta \leqq \mathfrak{F}/\sigma$. Then by the canonical homomorphism $\mathfrak{F}/\theta \to \mathfrak{F}/\sigma$, we obtain $I(\mathfrak{F}/\theta) \subseteq I(\mathfrak{F}/\sigma)$ and consequently, $KI(\mathfrak{F}/\theta) \supseteq KI(\mathfrak{F}/\sigma)$. Conversely, suppose $KI(\mathfrak{F}/\theta) \supseteq KI(\mathfrak{F}/\sigma)$. Consider a free basis v_i $(i \in I)$ for the system $\mathfrak{F}$. Since $\mathfrak{F}/\theta$ is a fully characteristic system, the collection $\{v_i\theta \mid i \in I\}$ is a free basis. By hypothesis, $\mathfrak{F}/\sigma$ belongs to

*) That is, when $\theta \subseteq \sigma$.

the variety generated by the system $\mathfrak{F}/\theta$. Therefore, the mapping

$$\varphi\colon v_i\theta \to v_i\sigma \qquad (i \in I) \tag{9}$$

can be extended to a homomorphism $\varphi\colon \mathfrak{F}/\theta \to \mathfrak{F}/\sigma$, the relations (9) showing that φ coincides with the homomorphism $\varepsilon\colon \mathfrak{F}/\theta \to \mathfrak{F}/\sigma$; this means that $\mathfrak{F}/\theta \leq \mathfrak{F}/\sigma$.

Thus, we have proved the following basic

Theorem 2. *Let $\mathfrak{F}$ be a free system of countable rank in the variety $\mathfrak{M}$. The mapping* (1) *of the partially ordered collection $X(\mathfrak{F})$ of all fully characteristic quotient systems of $\mathfrak{F}$ onto the partially ordered collection $M(\mathfrak{M})$ of all subvarieties of the variety $\mathfrak{M}$ is an anti-isomorphism of $X(\mathfrak{F})$ onto $M(\mathfrak{M})$.*

It was remarked above that the collection $M(\mathfrak{M})$ has the lattice property and the corresponding lattice $M(\mathfrak{M})$ is complete.

From Theorem 2, it follows that the collection $X(\mathfrak{F})$ also has the lattice property and the lattices $X(\mathfrak{F})$ and $M(\mathfrak{M})$ are anti-isomorphic.

Whenever varieties of algebras are being considered, the quotient systems $\mathfrak{A}/\theta$ are uniquely determined by the corresponding congruences θ (Section 2.4). If the quotient algebra $\mathfrak{A}/\theta$ is characteristic or fully characteristic, then the congruence θ is called characteristic or fully characteristic, respectively. From the definitions of characteristic and fully characteristic quotient system, we immediately obtain

Corollary 3. *A congruence θ on an algebra $\mathfrak{A}$ is characteristic (fully characteristic) if and only if*

$$x\theta y \Rightarrow x\varphi\theta y\varphi \qquad (x, y \in \mathfrak{A}) \tag{10}$$

for any automorphism (endomorphism) $\varphi\colon \mathfrak{A} \to \mathfrak{A}$.

Necessity is obvious since (10) can be rewritten in the form

$$x\theta = y\theta \Rightarrow xy\theta = y\varphi\theta,$$

i.e. in the form of formula (2) with P coinciding with $=$. Suppose conversely, that (10) is true for some congruence θ and endomorphism φ. We must obtain the relation

$$F(a_1\theta, \ldots, a_n\theta) = a\theta \Rightarrow F(a_1\varphi\theta, \ldots, a_n\varphi\theta) = a\varphi\theta \qquad (F \in \Omega).$$

But from $F(a_1\theta, \ldots, a_n\theta) = a\theta$ follows $F(a_1, \ldots, a_n)\theta = a\theta$. From this, using (10), we conclude that $F(a_1, \ldots, a_n)\varphi\theta = a\varphi\theta$ and therefore $F(a_1\varphi\theta, \ldots, a_n\varphi\theta) = a\varphi\theta$.

In terms of the theory of congruences, Theorem 2 may be reformulated now in this form:

Corollary 4. *Suppose $\mathfrak{F}$ is a free algebra of countable rank in a variety of algebras $\mathfrak{M}$. Then the mapping*

$$\theta \to KI(\mathfrak{F}/\theta)$$

is an anti-isomorphism of the lattice of fully characteristic congruences of the algebra $\mathfrak{F}$ onto the lattice of all subvarieties of the variety $\mathfrak{M}$.

To prove this, we merely must note that

$$\mathfrak{A}/\theta \leqq \mathfrak{A}/\sigma \Leftrightarrow \theta \subseteq \sigma$$

for any congruences θ and σ on an arbitrary algebra $\mathfrak{A}$.

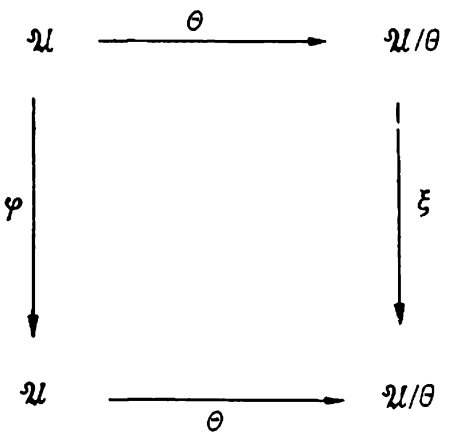

Fig. 1

By Section 11.3, a congruence θ on an algebra $\mathfrak{A}$ is *quasiverbal* (*verbal*) if the canonical homomorphism $\mathfrak{A} \to \mathfrak{A}/\theta$ is a $\mathfrak{K}$-morphism for a suitable quasivariety (variety) $\mathfrak{K}$. It is clear that here "suitable" can be replaced by "generated by $\mathfrak{A}$". From the properties proved above for fully characteristic congruences, it follows that the concepts "verbal" and "fully characteristic" are equivalent for algebras free in themselves. In general, we have

Theorem 5. *Every quasiverbal quotient system $\mathfrak{A}/\theta$ of an arbitrary system $\mathfrak{A}$ is fully characteristic.*

Suppose φ is an endomorphism of $\mathfrak{A}$. Since by hypothesis, the mapping $\theta\colon \mathfrak{A} \to \mathfrak{A}/\theta$ is a $\mathfrak{K}$-morphism, there is an endomorphism ξ of $\mathfrak{A}/\theta$ such that the diagram in Fig. 1 is commutative. From this, we obtain $P(a_1\theta, \ldots, a_n\theta) \Rightarrow P(a_1\theta\xi, \ldots, a_n\theta\xi) \Rightarrow P(a_1\varphi\theta, \ldots, a_n\varphi\theta)$, where $P \in \{\Omega, =\}$ and $a_1, \ldots, a_n$ are arbitrary elements of $\mathfrak{A}$.

It is easy to see that a quasiverbal congruence need not be verbal. For example, suppose $\mathfrak{K}$ is the quasivariety of all abelian torsion-free groups and $\mathfrak{A} = \mathfrak{B} \times \mathfrak{B}_p$, where $\mathfrak{B}$ is a free abelian group of countable rank and $\mathfrak{B}_p$ is cyclic of finite order $p > 1$. It is clear that $\mathfrak{B}_p$ is the minimal invariant subgroup in $\mathfrak{A}$ which yields a quotient group belonging to $\mathfrak{K}$. Consequently, $\mathfrak{A}/\mathfrak{B}_p$ is quasiverbal. If it were verbal, then it would belong to some variety $\mathfrak{L}$. This variety would contain all the quotient groups of $\mathfrak{B}$ including the group $\mathfrak{A}$ itself. This would mean that the mapping $\mathfrak{A} \to \mathfrak{A}/\mathfrak{B}_p$ would not be an $\mathfrak{L}$-morphism.

14.3 Minimal Varieties and Quasivarieties. A collection $\mathfrak{S}$ of identities (quasi-identities) of given signature Ω is called *equationally complete* or *I-complete* (*quasi-equationally complete* or *Q-complete*) if all formulas in $\mathfrak{S}$ are true in some nontrivial Ω-system and any other identity (quasi-identity) of signature Ω either is a formal consequence of the collection $\mathfrak{S}$ or together with $\mathfrak{S}$ forms a collection of true formulas only in the trivial Ω-system.

A variety (quasivariety) $\mathfrak{K}$ of signature Ω is called *minimal* or *atomic* if $\mathfrak{K}$ is an atom in the lattice of all varieties (quasivarieties) of signature Ω, i.e. if $\mathfrak{K}$ contains a nontrivial system, but within $\mathfrak{K}$ there are no subvarieties (subquasivarieties) distinct from $\mathfrak{K}$ or the trivial system.

Comparing the definitions of minimality and completeness given above, we come to the conclusion that *a collection of identities (quasi-identities) is I-complete (Q-complete) if and only if this collection defines a minimal variety (minimal quasivariety).*

An algebraic system of signature Ω is called I-complete (Q-complete) if the collection of all identities (quasi-identities) which hold in that system is I-complete (Q-complete), i.e. if it generates a minimal variety (quasivariety).

These definitions immediately imply (cf. Section 13.1)

Corollary 1. *For any nontrivial variety (quasivariety) $\mathfrak{K}$, the following statements are equivalent:*

a) *$\mathfrak{K}$ is a minimal variety (quasivariety);*

b) *$\mathfrak{K}$ is generated (as a variety or quasivariety, respectively) by any of its nontrivial systems;*

c) *all nontrivial systems in $\mathfrak{K}$ are I-equivalent (Q-equivalent).*

The proof is obvious. We point out a special case of this corollary.

Corollary 2. *Every minimal variety and minimal quasivariety is generated by its free system of rank 2.*

Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are free systems of rank 2 in the minimal variety $\mathfrak{K}$ and the minimal quasivariety $\mathfrak{L}$, respectively. Since the classes $\mathfrak{K}$ and $\mathfrak{L}$

are nontrivial, the systems $\mathfrak{F}$ and $\mathfrak{G}$ are also nontrivial (Section 12.2). But then the variety $KI\mathfrak{F}$ and quasivariety $KQ\mathfrak{G}$ will also be nontrivial. From the inclusions

$$KI\mathfrak{F} \subseteq \mathfrak{K}, \qquad KQ\mathfrak{G} \subseteq \mathfrak{L}$$

and the minimality of $\mathfrak{K}$ and $\mathfrak{L}$, it follows that

$$KI\mathfrak{F} = \mathfrak{K}, \qquad KQ\mathfrak{G} = \mathfrak{L}.$$

In Corollary 2, free systems of rank 2 can not be replaced by free systems of rank 1 since the nontriviality of class $\mathfrak{K}$ does not always imply the nontriviality of a free system of rank 1.

Corollary 3. *Every minimal variety of fixed signature contains a unique minimal quasivariety of the same signature. There exist minimal quasivarieties not contained in any minimal variety of given signature.*

Suppose $\mathfrak{M}$ is a minimal variety of fixed signature and $\mathfrak{F}_2$ is an $\mathfrak{M}$-free system of rank 2. Every minimal quasivariety $\mathfrak{K}$ contained in $\mathfrak{M}$ is generated by its free system $\mathfrak{G}_2$ of rank 2. The system $\mathfrak{G}_2$ generates a certain subvariety $\hat{\mathfrak{G}}_2$ of the variety $\mathfrak{M}$ and $\mathfrak{G}_2$ is free in $\hat{\mathfrak{G}}_2$. Since the variety $\mathfrak{M}$ is minimal, $\hat{\mathfrak{G}}_2 = \mathfrak{M}$. Therefore, $\mathfrak{F}_2$ and $\mathfrak{G}_2$ are isomorphic systems and

$$\mathfrak{K} = KQ\mathfrak{G}_2 = KQ\mathfrak{F}_2.$$

The converse argument does not hold. Indeed, let $\mathfrak{K}$ be a minimal quasivariety generated by the free system $\mathfrak{G}_2$ of rank 2. Consider the variety $\mathfrak{M} = \hat{\mathfrak{G}}_2$ generated by the system $\mathfrak{G}_2$. Since the quasivariety $\mathfrak{K}$ is not necessarily homomorphically closed, the system $\mathfrak{G}_2$ may have within it a free homomorphic image $\mathfrak{F}_2$ not belonging to $\mathfrak{K}$. Then

$$\hat{\mathfrak{F}}_2 \subseteq \hat{\mathfrak{G}}_2, \qquad \mathfrak{K} \cap \hat{\mathfrak{F}}_2 = \{\mathfrak{E}\},$$

where $\mathfrak{E}$ is the trivial system, and hence $\hat{\mathfrak{F}}_2 \subset \hat{\mathfrak{G}}_2$.

As an example, consider the quasivariety $\mathfrak{K}$ whose signature consists of one unary operator symbol f and which is defined by the quasi-identities

$$f^2x = x; \qquad fx = x \to x = y.$$

The free $\mathfrak{K}$-algebra $\mathfrak{A}$ with generators a and b obviously contains 4 elements: a, fa, b, fb. What quotient algebras does the algebra $\mathfrak{A}$ have? Suppose σ is a congruence on $\mathfrak{A}$. If $fa\sigma b$, then $f^2a\ \sigma fb$ and $a\sigma fb$. Hence $\mathfrak{A}/\sigma$ is either trivial or is the $\mathfrak{K}$-free algebra with the generator $a\sigma$. Matters are the same if $fa\sigma fb$, and hence

$$f^2a\sigma f^2b,\ a\sigma b.$$

If $a\sigma fa$, then in the quotient algebra $\mathfrak{A}/\sigma$, we have $f(a\sigma) = a\sigma$, and this means either $\mathfrak{A}/\sigma \notin \mathfrak{K}$ or $\mathfrak{A}/\sigma$ is the trivial algebra.

Thus, there are no algebras of rank 2 in $\mathfrak{K}$ free in themselves and distinct from $\mathfrak{A}$. Therefore, the quasivariety $\mathfrak{K}$ is minimal.

It is clear that no identity of the form $f^m x = f^n y$ is satisfied in $\mathfrak{A}$. If an identity of the form $f^m x = f^n y$ holds in $\mathfrak{A}$, then the difference $m - n$ is obviously even; therefore, this identity is a formal consequence of the identity $f^2 x = x$. We see that the variety $\hat{\mathfrak{A}}$ generated by the system $\mathfrak{A}$ is characterized by the single identity $f^2 x = x$. However, the variety $\hat{\mathfrak{A}}$ is not minimal since it contains the subvariety $\mathfrak{M}_0$ characterized by the identity $fx = x$. The latter identity does not hold in $\mathfrak{A}$ and hence $\mathfrak{M}_0 \neq \hat{\mathfrak{A}}$.

Somewhat more specific to the theory of varieties is

Theorem 4. *A nontrivial variety is minimal if and only if each of its nontrivial systems is I-complete.*

All nontrivial systems of a minimal quasivariety are Q-complete. If in a nontrivial quasivariety $\mathfrak{K}$, compositions are injective and all nontrivial systems in $\mathfrak{K}$ are Q-complete, then the quasivariety $\mathfrak{K}$ is minimal.

Suppose that $\mathfrak{K}$ is a minimal variety and let $\mathfrak{A}$ be a nontrivial system of $\mathfrak{K}$. Since

$$\mathfrak{A} \in KI\mathfrak{A} \subseteq \mathfrak{K}$$

and $\mathfrak{K}$ is minimal, we have $KI\mathfrak{A} = \mathfrak{K}$ and consequently, the system $\mathfrak{A}$ is I-complete. The same argument holds also for quasivarieties.

Conversely, suppose that in a nontrivial variety $\mathfrak{K}$, all nontrivial systems are I-complete. Consider nontrivial systems $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathfrak{K}$. Since the systems $\mathfrak{A}$ and $\mathfrak{B}$ are homomorphic images of $\mathfrak{A} \times \mathfrak{B}$, we have

$$I(\mathfrak{A} \times \mathfrak{B}) \subseteq I\mathfrak{A}, \quad I(\mathfrak{A} \times \mathfrak{B}) \subseteq I\mathfrak{B}.$$

But the systems $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{A} \times \mathfrak{B}$ are I-complete. Therefore,

$$I\mathfrak{A} = I(\mathfrak{A} \times \mathfrak{B}) = I\mathfrak{B},$$

i.e. all nontrivial $\mathfrak{K}$-systems are I-equivalent. By c) of Corollary 1, this implies the minimality of the variety $\mathfrak{K}$.

If $\mathfrak{K}$ is a quasivariety, then we must replace $\mathfrak{A} \times \mathfrak{B}$ in the above reasoning by the $\mathfrak{K}$-composition $\mathfrak{A} * \mathfrak{B}$ and note that, by hypothesis, the system $\mathfrak{A} * \mathfrak{B}$ contains $\mathfrak{A}$ and $\mathfrak{B}$ as subsystems.

The condition of injectiveness of compositions in Theorem 4 is essential. For example, consider systems whose signature consists of one-place predicate symbols P and Q. The quasivariety $\mathfrak{K}$ defined by the formulas

$$x = y, \qquad P(x) \to Q(x),$$

consists of three one-element models $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$, which have the respective diagrams

$$D(\mathfrak{A}_1) = \{P(a), Q(a)\},$$

$$D(\mathfrak{A}_2) = \{\neg P(a), \neg Q(a)\},$$

$$D(\mathfrak{A}_3) = \{\neg P(a), Q(a)\}.$$

The model $\mathfrak{A}_1$ is trivial and the model $\mathfrak{A}_2$ is absolutely free. The pair $\mathfrak{A}_1$ and $\mathfrak{A}_2$ forms a minimal quasivariety defined by the formulas

$$x = y, \qquad P(x) \to Q(x), \qquad Q(x) \to P(x),$$

and the pair $\mathfrak{A}_1$ and $\mathfrak{A}_3$ form a minimal quasivariety defined by the formulas

$$x = y, \qquad Q(x).$$

From this it follows that $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are Q-complete. Thus, all nontrivial models of the quasivariety $\mathfrak{K}$ are Q-complete, but the quasivariety $\mathfrak{K}$ itself is not minimal.

In Section 10.1, we presented a general principle for completing collections of first-order formulas. For varieties and quasivarieties, it leads to

Theorem 5. *Every nontrivial variety (quasivariety) $\mathfrak{K}$ of algebraic systems of arbitrary signature Ω contains at least one minimal variety (quasivariety).*

To the collection of formulas

$$\mathfrak{S} = I\mathfrak{K} \cup \exists xy(x \neq y)$$

we can adjoin various collections $\mathfrak{T}$ of identities of signature Ω such that the resulting collection $\mathfrak{S} \cup \mathfrak{T}$ is satisfiable. According to the above completion principle (Section 10.1, Corollary 2), there exists a maximal collection $\mathfrak{T}$ with the property that $\mathfrak{S} \cup \mathfrak{T}$ is satisfiable. Denote by $\mathfrak{K}_1$ the variety defined by the identities $I\mathfrak{K}$ and the maximal collection $\mathfrak{T}$. Since the collection $\mathfrak{S} \cup \mathfrak{T}$ is satisfiable, the variety $\mathfrak{K}_1$ is nontrivial. If $\mathfrak{K}_1$ does not contain any nontrivial one-element systems, then the variety $\mathfrak{K}_1$ is minimal, $\mathfrak{K}_1 \subseteq \mathfrak{K}$ and the assertion of Theorem 5 is true (for varieties).

Suppose that $\mathfrak{K}_1$ contains a nontrivial one-element system $\mathfrak{A}$. Then $\mathfrak{A}$ belongs to the variety $\mathfrak{K}_2$ defined by the identities

$$I\mathfrak{K}, \qquad \forall xy(x = y), \tag{1}$$

and therefore $\mathfrak{K}_2$ is a nontrivial subvariety of the variety $\mathfrak{K}$. The free system $\mathfrak{F}_1$ in $\mathfrak{K}_2$ will also be nontrivial and consequently, for a suitable predicate symbol $P \in \Omega$

$$P(a, \ldots, a) = \mathsf{F} \qquad (a \in \mathfrak{F}_1).$$

Suppose P_λ $(\lambda \in L)$ are the remaining predicate symbols in Ω. The variety $\mathfrak{K}_3$ defined by the identities (1) and the additional identities

$$P_\lambda(x, \ldots, x) = \mathsf{T} \qquad (\lambda \in L),$$

consists only of the trivial system and a one-element system $\mathfrak{B}$ whose diagram contains only one negative formula, namely, the formula $\neg P(a, \ldots, a)$. Therefore, the variety $\mathfrak{K}_3$ is minimal. Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{F}_1$, $\mathfrak{B} \in \mathfrak{K}_2$ and consequently, $\mathfrak{K} \supseteq \mathfrak{K}_2 \supseteq \mathfrak{K}_3$, as required.

The assertion of Theorem 5 for quasivarieties is proved similarly. To the collection of formulas

$$\mathfrak{S} = Q\mathfrak{K} \cup \exists xy(x \neq y)$$

we adjoin a maximal collection $\mathfrak{T}$ of quasi-identities such that the collection $\mathfrak{S} \cup \mathfrak{T}$ is satisfiable. Let $\mathfrak{K}_1$ denote the quasivariety defined by the quasi-identities $Q\mathfrak{K} \cup \mathfrak{T}$. If $\mathfrak{K}_1$ does not contain nontrivial one-element systems, then $\mathfrak{K}_1$ is the desired minimal subquasivariety in $\mathfrak{K}$. If $\mathfrak{K}_1$ contains some nontrivial one-element system $\mathfrak{A}$, then we consider the quasivariety $\mathfrak{K}_2$ defined by the formulas

$$\forall xy(x = y), \quad P_\lambda(x, \ldots, x) \to P_\mu(x, \ldots, x), \quad P_a(x, \ldots, x),$$

where P_λ and P_μ are all the predicate symbols of Ω whose values are false in $\mathfrak{A}$ and P_a are those predicate symbols whose values in $\mathfrak{A}$ are true. It is clear that $\mathfrak{K}_2$ consists only of the trivial system and the system $\mathfrak{A}$. Hence the quasivariety $\mathfrak{K}_2$ is minimal. It is the desired minimal subvariety in $\mathfrak{K}$.

Theorem 5 only speaks of the existence of a minimal subvariety in each variety. The number of minimal varieties contained in some given variety $\mathfrak{K}$ depends on $\mathfrak{K}$.

Theorem 6. *If all the systems of a variety (quasivariety) $\mathfrak{K}$ are locally finite and its signature Ω is finite, then $\mathfrak{K}$ contains only a finite number of minimal varieties (quasivarieties).*

According to Corollary 2, every minimal variety (quasivariety) in $\mathfrak{K}$

is generated by a $\mathfrak{K}$-system $\mathfrak{G}$ of rank 2, free in itself. Every system $\mathfrak{G}$ is a homomorphic image of a $\mathfrak{K}$-free system $\mathfrak{F}$ of rank 2. Since $\mathfrak{F}$ is generated by two elements, by hypothesis, the system $\mathfrak{F}$ is finite. Therefore, the number of distinct congruences on $\mathfrak{F}$ is also finite. Since the signature of $\mathfrak{F}$ contains only a finite set of symbols, the number of distinct homomorphic images of $\mathfrak{F}$ corresponding to a given congruence is also finite. Consequently, $\mathfrak{F}$ has only a finite number of distinct homomorphic images, among which are all the systems $\mathfrak{G}$ free in themselves. Therefore, the number of nonisomorphic systems $\mathfrak{G}$ is finite and hence so is the number of minimal subvarieties and subquasivarieties in $\mathfrak{K}$.

According to Section 14.1, every variety generated by a finite system consists only of locally finite systems. Therefore, Theorem 5 implies

Corollary 7 (D. Scott [64]). *If a variety $\mathfrak{M}$ is generated by a finite system and its signature is finite, then $\mathfrak{M}$ contains only a finite number of minimal varieties and minimal quasivarieties.*

It is natural to ask: How many distinct minimal varities and minimal quasivarieties are there of given signature Ω? Since every minimal variety contains a unique minimal quasivariety, the number of minimal varieties of given signature does not exceed the number of minimal quasivarieties. The question of the number of minimal varieties was first studied by Kalicki [24], who showed that there exists a continuum of distinct minimal varieties of groupoids. Let us prove the following statement using Kalicki's method.

Theorem 8. *There are at least $2^{|\Omega|}$ distinct minimal varieties of Ω-unoids. If the signature Ω of the algebras under consideration contains at least one non-unary operator symbol, then the number of distinct minimal varieties of signature Ω is equal to $2^{|\Omega|+\aleph_0}$.*

Suppose $\Omega = \{f_i \mid i \in I\}$ and K is an arbitrary subset of I. Consider the variety $\mathfrak{N}_K$ of Ω-unoids defined by the identities

$$\left.\begin{aligned} f_i x &= x \qquad (i \in K), \\ f_j x &= f_k y \qquad (j, k \in I - K). \end{aligned}\right\} \tag{2}$$

The variety $\mathfrak{N}_k$ is certainly nontrivial since it contains the two-element algebra $\langle \{a, b\}, \Omega \rangle$ in which

$$f_i(x) = x, \quad f_j(a) = f_j(b) = b \qquad (i \in K, \quad j \in I - K).$$

Any identity of signature Ω has the form a), b), c) or d) (see Section 13.3) and it is easy to check that either it is a formal consequence of the

collection (2), or together with (2) forms a collection of formulas which implies the identity $x = y$. Consequently, the variety $\mathfrak{N}_k$ is minimal.

Thus, the number of minimal varieties of unoids of signature Ω is not less than the number of distinct subsets K of I, which is $2^{|\Omega|}$.

Now consider the case where the signature $\Omega = \{f_i \mid i \in I\}$ is infinite and contains at least one non-unary operator symbol. In Section 13.3, it was shown that the lattice of subvarieties of a variety of Ω-algebras satisfying the identities

$$f_i(x, x_2, \ldots, x_{n_i}) = f_i(x, y_2, \ldots, y_{n_i}) \qquad (i \in I)$$

is isomorphic to the lattice of all varieties of signature $\Omega^* = \{g_i \mid i \in I\}$, where the g_i are one-place operator symbols. As was just proved, the cardinality of the collection of minimal varieties of Ω^*-unoids is at least $2^{|\Omega|}$. Therefore, there exist at least $2^{|\Omega|}$ minimal varieties of signature Ω. Since there are altogether $2^{|\Omega|+\aleph_0} = 2^{|\Omega|}$ varieties of signature Ω, the cardinality of the collection of minimal Ω-varieties is precisely $2^{|\Omega|+\aleph_0}$.

It remains to consider the case of a finite signature Ω which contains at least one non-unary operator symbol f_α. Consider the subvarieties of the variety given by the identities

$$f_\alpha(x, y, x_3, \ldots, x_{n_\alpha}) = f_\alpha(x, y, z_3, \ldots, z_{n_\alpha}),$$

$$f_i(x_1, \ldots, x_{n_i}) = x_1 \qquad (i \neq \alpha, \quad i \in I),$$

we see that we are reduced to proving the assertion of Theorem 8 only for varieties of groupoids with one binary operation which we shall denote by $\circ$. Following Kalicki [24], we introduce the abbreviations

$$x \circ x = 2x, \qquad 2^n x \circ 2^n x = 2^{n+1} x \qquad (n = 1, 2, \ldots).$$

Let M be any nonempty set of natural numbers greater than one. Let M' be the remaining natural numbers greater than one. Denote by $\mathfrak{A}_M$ the variety of $\circ$-groupoids defined by the collection of identities

(i) $2x \circ x = 2y \circ y$,

(ii) $2^m x \circ x = 2x \circ x \qquad (m \in M)$,

(iii) $2^n x \circ x = x \qquad (n \in M')$,

(iv) $x \circ y = y \circ x$.

Let us show that every variety $\mathfrak{A}_M$ is nontrivial. To this end, we define the operation $\circ$ on the set N of natural numbers, putting for any

natural number i

$$(i+1)\circ i = i\circ(i+1) = 0,$$

$$(i+m)\circ i = i\circ(i+m) = 0 \qquad (m\in M),$$

$$(i+n)\circ i = i\circ(i+n) = i \qquad (n\in M'),$$

$$i\circ i = i+1.$$

It is clear that the groupoid $\langle N, \circ\rangle$ thus defined belongs to the variety $\mathfrak{A}_M$, and hence this variety is nontrivial.

It is easy to see also that the intersection of any two varieties $\mathfrak{A}_M$ and $\mathfrak{A}_L$ corresponding to distinct sets M and L contains only the trivial groupoid. Indeed, if $M \neq L$, then there exists a number s, belonging to $(M \cup L') \cup (M' \cup L)$. Therefore, in an arbitrary groupoid $\langle A, \circ\rangle$, the following identities are simultaneously satisfied in the intersection of the varieties $\mathfrak{A}_M$ and $\mathfrak{A}_L$:

$$2x\circ x = 2y\circ y,$$

$$2^s x\circ x = 2x\circ x,$$

$$2^s x\circ x = x.$$

These imply the identity $x = y$.

According to Theorem 5, every variety $\mathfrak{A}_M$ has some minimal subvariety $\mathfrak{A}_M{}^*$. Since $\mathfrak{A}_M \cap \mathfrak{A}_L = \{\mathfrak{E}\}$ for $M \neq L$, we have $\mathfrak{A}_M{}^* \neq \mathfrak{A}_L{}^*$. Therefore, the cardinality of the collection of the varieties $\mathfrak{A}_M{}^*$ coincides with the cardinality of the collection of all nonempty subsets of the set of natural numbers, i.e. the cardinality of the continuum.

Thus, the cardinality of the collection of all minimal varieties of groupoids is not less than the cardinality of the continuum. Since the cardinality of the collection of all varieties of groupoids is less than or equal to $\aleph$, the collection of all minimal varieties of groupoids is equal to the cardinality of the continuum.

Since the identities (i)—(iv) define a variety of *commutative* groupoids, the above reasoning actually proves the stronger

Corollary 9. *The collection of all minimal varieties of commutative groupoids has the cardinality of the continuum.*

The problem of determining the cardinality of the collection of minimal subvarieties contained in some fixed variety $\mathfrak{M}$ has been solved for a number of important cases.

Certain other results on minimal varieties of systems are given in the complements to this section.

Problems and Complements

1. A variety of groups is minimal if and only if it is abelian of some prime exponent (i.e. consists of abelian groups satisfying the identity $x^p = e$, where p is a fixed prime).

2. A variety of semigroups is minimal if and only if it can be presented in the class of all semigroups by one of the following systems of identities:

$$xy = yx, \quad xx = x,$$
$$xy = x,$$
$$xy = y,$$
$$xy = zw,$$
$$xy = yx, \quad x^p y = y \quad (p, \text{ a prime})$$

(Kalicki and Scott [25]).

3. Distributive lattices constitute the only minimal variety of lattices (Kalicki and Scott [25]).

4. A variety $\mathfrak{K}$ of associative rings is minimal if and only if it is either the variety of p-zero-rings, or the variety of p-rings (p, a prime), i.e. if $\mathfrak{K}$ can be defined in the class of associative rings either by the identities $px = 0 = xy$, or by the identities $px = 0$, $x^p = x$ (Tarski [71]).

Concerning minimal quasivarieties of associative rings, see Vinogradov [24].

5. The set of minimal varieties of totally symmetric quasigroups has the cardinality of the continuum (Bol'bot [9]).

6. Suppose m and n are fixed integers satisfying $1 \leqq m \leqq n$. As in Section 12.2, denote by $\mathfrak{S}_{m,n}$ the variety of algebras $\langle A, F_1, \ldots, F_n, G_1, \ldots, G_m \rangle$ of the type $\langle m, \ldots, m, n, \ldots, n \rangle$ defined by the identities

$$F_i(G_1(x_1, \ldots, x_n), \ldots, G_m(x_1, \ldots, x_n)) = x_i \quad (i = 1, \ldots, n),$$
$$G_j(F_1(x_1, \ldots, x_m), \ldots, F_n(x_1, \ldots, x_m)) = x_j \quad (j = 1, \ldots, m).$$

Among the varieties $\mathfrak{S}_{m,n}$ $(1 \leqq m \leqq n)$ the only minimal ones are the varieties $\mathfrak{S}_{1,n}$ $(n > 1)$. If $n \geqq m \geqq 2$, then the variety $\mathfrak{S}_{m,n}$ contains a continuum of minimal subvarieties. The lattice $M(\mathfrak{S}_{1,1})$ of subvarieties of the variety $\mathfrak{S}_{1,1}$ is isomorphic to the lattice obtained from the lattice of integers with divisibility ordering $(x \leqq y \Leftrightarrow x \mid y)$ by adjoining a null and maximal element (Akataev and Smirnov[2]).

7. Let $\mathfrak{K}$ be a fixed class of algebraic systems and $\mathfrak{U}$ and $\mathfrak{V}$ subclasses of $\mathfrak{K}$. Denote by $\mathfrak{U} \underset{\mathfrak{K}}{\circ} \mathfrak{V}$ the class of all those $\mathfrak{K}$-systems $\mathfrak{A}$ each of which has a congruence θ such that the quotient system $\mathfrak{A}/\theta$ belongs to $\mathfrak{V}$, and all the congruence classes $a\theta$ $(a \in \mathfrak{A})$ which are $\mathfrak{K}$-systems belong to $\mathfrak{U}$:

$$\mathfrak{A} \in \mathfrak{U} \underset{\mathfrak{K}}{\circ} \mathfrak{V} \Leftrightarrow \mathfrak{A} \in \mathfrak{K} \;\&\; (\exists\, \theta)\, (\mathfrak{A}/\theta \in \mathfrak{V}) \;\&\; (\forall\, a)\, a\theta \in \mathfrak{K} \to a\theta \in \mathfrak{U}.$$

If $\mathfrak{K}$ is a quasivariety of algebraic systems whose signature contains only a finite number of function symbols, then the product $\mathfrak{U} \underset{\mathfrak{K}}{\circ} \mathfrak{V}$ of any two subquasivarieties $\mathfrak{U}$, $\mathfrak{V}$ of $\mathfrak{K}$ is a quasivariety. In particular, we can speak of the groupoid $G_Q\mathfrak{K}$ of subquasivarieties of any quasivariety $\mathfrak{K}$ of universal algebras of finite signature (Mal'cev [49]).

8. If $\mathfrak{K}$ is the variety of all groups, and $\mathfrak{U}$, $\mathfrak{V}$ are subvarieties of $\mathfrak{K}$, then the product $\mathfrak{U} \underset{\mathfrak{K}}{\circ} \mathfrak{V}$ is a variety and coincides with the product $\mathfrak{U}\mathfrak{V}$ in the sense of H. Neumann [55].

9. Let $\mathfrak{K}$ be the variety of all semigroups with distinguished identity e and let $\mathfrak{A}$ be the subvariety of all commutative semigroups in $\mathfrak{K}$. In [49], it is shown that the product $\mathfrak{A} \underset{\mathfrak{K}}{\circ} \mathfrak{A}$ is not a variety. Indeed, let $\mathfrak{F}$ be a semigroup free in $\mathfrak{K}$ with free basis a, b. Consider the congruence θ:

$$a^{m_1}b^{n_1} \dots a^{m_k}b^{n_k}\theta a^{p_1}b^{q_1} \dots a^{p_l}b^{q_l} = m_1 + \dots + m_k = p_1 + \dots + p_l.$$

$\mathfrak{F}/\theta$ is a commutative semigroup, and hence $\mathfrak{F}/\theta \in \mathfrak{A}$. Since e is a distinguished element in $\mathfrak{K}$, only those congruence classes $x\theta$ which contain e belong to $\mathfrak{K}$. There is only one such class: $e\theta = \{e, b, b^2, \dots\}$ and it is a commutative semigroup. Therefore $\mathfrak{F} \in \mathfrak{A} \underset{\mathfrak{K}}{\circ} \mathfrak{A}$. If the class $\mathfrak{A} \underset{\mathfrak{K}}{\circ} \mathfrak{A}$ were a variety, then it would also contain all the quotients of $\mathfrak{F}$. The symmetric group of order 5, $\mathfrak{C}_5$, is generated by the two elements (12), (12345) and is therefore isomorphic to a quotient semigroup of $\mathfrak{F}$. At the same time, the group $\mathfrak{C}_5 \notin \mathfrak{A} \underset{\mathfrak{K}}{\circ} \mathfrak{A}$ since it has no nontrivial abelian normal divisors. Thus, $\mathfrak{A} \underset{\mathfrak{K}}{\circ} \mathfrak{A}$ is not a variety.

10. A *polar* operation in an algebraic system $\langle A, \Omega \rangle$ is a unary term operation $e(x)$ (i.e. expressible as a term) satisfying

$$(\forall\, xy \in A) \quad e(x) = e(y) = e, \qquad F(e, \dots, e) = e \quad (F \in \Omega_F).$$

The element e is called a *pole* of the system $\langle A, \Omega \rangle$. A *polar* of a class $\mathfrak{K}$ is a unary term which represents a polar operation in every $\mathfrak{K}$-system. A class of systems which has at least one polar is called *polarized*. If $\mathfrak{K}$ is a polarized variety of algebra and if congruences are permutable in all $\mathfrak{K}$-algebras, then the $\mathfrak{K}$-product of any two $\mathfrak{K}$-subvarieties $\mathfrak{U}$, $\mathfrak{V}$ is a variety. Thus, for a polarized variety of $\mathfrak{K}$-algebras with permutable congruences, one can speak of the groupoid $G_I\,\mathfrak{K}$ of subvarieties. For instance, one can consider the groupoid of subvarieties of any variety of groups, loops, rings, etc. (Mal'cev [49]).

11. If $\mathfrak{K}$ is the variety of all groups, then $G_I\,\mathfrak{K}$ is a free semigroup with zero and identity (Šmel'kin [66], B. H. Neumann, H. Neumann, P. M. Neumann [54]).

The semigroup of varieties of Lie algebras over a field of characteristic zero is also free (Parfenov [57]).

12. In the variety of algebras $\langle A, F \rangle$ of type $\langle 3 \rangle$ defined by the identities

$$F(x, x, z) = F(x, z, x) = F(z, x, x) = z,$$

$$F\big(F(x_1, y_1, z_1), F(x_2, y_2, z_2), F(x_3, y_3, z_3)\big) = F\big(F(x_1, x_2, x_3), F(y_1, y_2, y_3), F(z_1, z_2, z_3)\big)$$

all algebras are free. The analogous property holds for the variety of quasigroups described by the identities: $y \cdot yx = x = xy \cdot y$, $xx = x$, $xy \cdot zw = xz \cdot yw$ (Aliev [3]).

13. The lattice of subvarieties of the variety of all modular lattices $\langle L, +, \cdot \rangle$ has the cardinality of the continuum (Baker [6]).

Bibliography

(Starred items are in Russian)

*1. Ado, I. D.: On nilpotent algebras and p-groups. Dokl. Akad. Nauk SSSR **40**, 339–342 (1943).

*2. Akataev, A. A., Smirnov, D. M.: Lattices of subvarieties of varieties of algebras. Algebra and Logic (Seminar) **7**, No. 1, 5–25 (1968).

*3. Aliev, I. Š.: On the minimal variety of a symmetric algebra. Algebra and Logic (Seminar) **5**, No 6, 5–14 (1966).

*4. Almagambetov, Ž. A.: On classes of axioms closed under given reduced products and powers. Algebra and Logic (Seminar) **4**, No. 3, 71–77 (1965).

5. Baker, K. A.: Equational classes of modular lattices. Pacific J. Math. **28**, No. 1, 9–15 (1969).

*6. Baranovič, T. M.: Free decompositions in intersections of primitive classes of algebras. Mat. Sb. **67**, 135–153 (1965).

7. Birkhoff, G.: On the structure of abstract algebras. Proc. Cambridge Philos. Soc. **31**, 433–454 (1935).

8. Birkhoff, G.: Lattice Theory. American Math. Soc., New York 1940.

*9. Bol'bot, A. D.: On equationally complete varieties of totally symmetric quasigroups. Algebra and Logic (Seminar) **6**, No. 2, 13–19 (1967).

10. Chang, C. C., Morel, A. C.: On closure under direct product. J. Symb. Logic **23**, 149–154 (1958).

11. Church, A.: Introduction to Mathematical Logic I. Princeton, New Jersey 1956.

12. Clifford, A. H., Preston, G. B.: The Algebraic Theory of Semigroups. American Math. Soc., Providence, R. I., Vol. **1**, 1961; Vol. **2**, 1966.

13. Cohn, P. M.: Universal Algebra. Harper and Row, Publishers, N. Y. 1965.

14. Feferman, S., Vaught, R. L.: The first-order properties of products of algebraic systems. Fund. Math. **47**, 57–103 (1959).

15. Frayne, T. E., Morel, A. C., Scott, D.: Reduced direct products. Fund. Math. **51**, 195–226 (1962).

16. Fujiwara, T.: Note on the isomorphism problem for free algebraic systems. Proc. Japan Acad. **31**, 135–136 (1955).

17. Hall, M.: The word problem for semigroups with two generators. J. Symb. Logic **14**, 115–119 (1949).

18. Hausdorff, F.: Set Theory. Chelsea Publishing CO., New York 1957.

19. Henkin, L.: Some interconnections between modern algebra and mathematical logic. Trans. Amer. Math. Soc. **74**, 410–427 (1953).

20. Higman, G.: Some remarks on varieties of groups. Quart. J. Math. Oxford Ser. **10**, 165–178 (1959).

21. Hilbert, D., Ackermann, W.: Principles of Mathematical Logic, Chelsea Publishing CO., New York 1950. English trans. of 2nd ed.

22. Horn, A.: On sentences which are true of direct unions of algebras. J. Symb. Logic **16**, 14–21 (1951).

23. Jónsson, B., Tarski, A.: On two properties of free algebras. Math. Scand. **9**, 95–101 (1961).
24. Kalicki, J.: The number of equationally complete classes of equations. Indag Math. **17**, 660–662 (1955).
25. Kalicki, J., Scott, D.: Equational completeness in abstract algebras. Indag. Math. **17**, 650–659 (1955).
26. Keisler, H. J.: Ultraproducts and elementary classes. Indag. Math. **23**, 477–495 (1961).
27. Keisler, H. J.: Reduced products and Horn classes. Trans. Amer. Math. Soc. **117**, 307–328 (1965).
28. Kleene, S. C.: Introduction to Metamathematics. Van Nostrand, Princeton New Jersey 1952.
29. Kochen, S.: Ultraproducts in the theory of models. Ann. of Math., Ser. 2, **74**, 221–261 (1961).
30. Kuroš, A. G.: Lectures on General Algebra. Chelsea Publishing Company, N. Y. 1963.
31. Kuroš, A. G.: Theory of Groups. 2 Vols., Chelsea Publishing CO., N.Y., 2nd ed. 1959–1960.
32. Lambek, J.: The immersibility of a semigroup into a group. Canad. J. Math. **3**, 34–43 (1951).
*33. Linnik, Ju. V.: Quaternions and Cayley numbers. Uspekhi Mat. Nauk **4**, No. 5, 49–65 (1949).
34. Łos, J.: Quelques rémarques, théorèmes et problèmes sur les classes définissables d'algèbres. Math. Interpretation of Formal Systems, Amsterdam, 98–113 (1955).
35. Lyndon, R. C.: Properties preserved under homomorphism. Pacific J. Math. **9**, 143–154 (1959).
36. Lyndon, R. C.: Properties preserved under subdirect products. Pacific J. Math. **9**, 155–164 (1959).
37. Machover, M.: A note on sentences preserved under direct products and powers. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **8**, 519–523 (1960).
*38. Mal'cev, A. I.: Untersuchungen aus dem Gebiete der mathematischen Logik. Mat. Sb. **1** (43), 323–336 (1936).
39. Mal'cev, A. I.: On the immersion of an algebraic ring into a field. Math. Ann. **113**, 686–691 (1937).
*40. Mal'cev, A. I.: On the embedding of associative systems into groups. I, Mat. Sb. **6**, 331–336 (1939).
*41. Mal'cev, A. I.: On the embedding of associative systems into groups. II, Mat. Sb., **8**, 251–264 (1940).
*42. Mal'cev, A. I.: On a general method for obtaining local theorems in group theory. Učen. Zap. Ivanovsk. Ped. Inst. Fiz-Mat. Fak. **1**, No. 1, 3–9 (1941).
*43. Mal'cev, A. I.: Algebraic systems. Transactions of Third All-Union Mathematics Conference, Moscow, **2**, 8 (1958).
*44. Mal'cev, A. I.: Axiomatizable classes of locally free algebras of certain types. Sibirsk. Mat. Ž. **3**, 729–743 (1962).
*45. Mal'cev, A. I.: Some questions in the theory of model classes. Transactions of Fourth All-Union Mathematics Conference, Leningrad, **1**, 169–198 (1963).
*46. Mal'cev, A. I.: Several remarks on quasivarieties of algebraic systems. Algebra and Logic (Seminar) **5**, No. 3, 3–9 (1966).

*47. Mal'cev, A. I.: Identical relations on varieties of quasigroups. Mat. Sb. **69**, 3—12 (1966).
*48. Mal'cev, A. I.: On some questions bordering algebra and mathematical logic. International Congress of Mathematicians, Abstracts of Invited Reports, Moscow, 26 (1966).
*49. Mal'cev, A. I.: On multiplication of classes of algebraic systems. Sibirsk. Mat. Ž. **8**, No. 2, 346—365 (1967).
50. Marczewski, E.: Independence and homomorphisms in abstract algebras. Fund. Math. **50**, 45—61 (1961).
51. Morley, M.: Categoricity in power. Trans. Amer. Math. Soc. **114**, 514—538 (1965).
52. Moufang, R.: Alternativkörper und der Satz von vollständigen Vierseit. Abh. Math. Sem. Univ. Hamburg **9**, Noes. 3—4 (1933).
53. Neumann, B. H.: Some remarks on infinite groups. J. London Math. Soc. **12**, 120—127 (1937).
54. Neumann, B. H., Neumann, H., Neumann, P. M.: Wreath products and varieties of groups. Math. Z. **80**, 44—62 (1962).
55. Neumann, H.: Varieties of Groups. Berlin—Heidelberg—New York: Springer 1967.
*56. Novikov, P. S.: Elements of Mathematical Logic. Fizmatgiz, Moscow 1959.
*57. Parfenov, V. A.: On varieties of Lie algebras. Algebra and Logic (Seminar) **6**, No. 4, 61—73 (1967).
58. Riguet, J.: Relations binaires, fermetures, correspondences de Galois. Bull. Soc. Math. France **76**, 1—4 (1948).
59. Robinson, A.: On the Metamathematics of Algebra. North-Holland, Amsterdam 1951.
60. Robinson, A.: Complete Theories. North-Nolland, Amsterdam 1956.
61. Robinson, A.: Introduction to Model Theory and the Metamathematics of Algebra. North-Holland, Amsterdam 1963.
62. Robinson, A.: Recent developments in model theory. Proc. of the International Congress of Logic, Methodology and Philosophy of Science (Stanford 1960), 60—70 (1962).
*63. Schmidt, O. Ju.: Infinite solvable groups. Mat. Sb. **17**, 145—162 (1945)
64. Scott, D.: Equationally complete extensions of finite algebras. Indag. Math. **18**, 35—38 (1956).
*65. Širšov, A. I.: Some questions in the theory of almost associative rings. Uspehi Mat. Nauk **13**, No. 6, 3—20 (1958).
*66. Šmel'kin, A. L.: Semigroup of varieties of groups. Dokl. Akad. Nauk SSSR **149**, 543—545 (1963).
67. Soundararajan, T.: On the automorphisms of the complex number field. Math. Magazine **40**, No. 4, 213 (1967).
68. Swierczkowski, S.: On isomorphic free algebras. Fund, Math. **50**, 35—44: (1961).
69. Tamura, T.: Attainability of systems of identities on semigroups. J. of Algebra **3**, No. 3, 261—276 (1966).
70. Tarski, A.: Contributions to the theory of models. Indag. Math. **16**, 572—588 (1954); **17**, 56—64 (1955).
71. Tarski, A.: Equationally complete rings and relation algebras. Indag. Math. **18**, 39—46 (1956).

72. Tarski, A., Vaught, R. L.: Arithmetical extensions of relational systems. Compositio Math. **13**, 81—102 (1957).
73. Vaught, R. L.: Applications of the Löwenheim-Skolem-Tarski theorem to problems of completeness and decidability. Indag. Math. **16**, 467—472 (1954).
74. Vinogradov, A. A.: Minimal quasivarieties of rings and relational algebras. Algebra and Logic (Seminar) **6**, No. 4, 3—10 (1967).
75. van der Waerden, B. L.: Modern Algebra. 2 Vols., Ungar, N.Y. 1949.
76. Zariski, O., Samuel, P.: Commutative Algebra. 2 Vols., Van Nostrand, Princeton, N.J. 1958, 1960.
77. Zorn, M.: Theorie der alternativen Ringe. Abh. Math. Sem. Hansischen Univ. **8**, 123—147 (1932).
78. Zorn, M.: Alternativkörper und quadratische Systeme. Abh. Sem. Hansischen Univ. **9**, 395—402 (1933).

Name Index

Ackermann, W. 128
Akataev, A. A. 303
Aliev, I. Š. 304
Almagambetov, Ž. A. 150, 152

Baker, K. A. 304
Baranovič, T. M. 266
Birkhoff, G. 267
Bol'bot, A. D. 303
Bokut', L. A. 80

Chang, C. C. 148, 169, 176
Cohen, P. J. 63
Cohn, P. M. 174, 217

Dedeking, R. 104
Dyck, W. 222

Feferman, S. 150
Frayne, T. E. 174, 176
Frobenius, G. 100
Fujiwara, T. 251, 253

Hall, M. 236
Hausdorff, F. 29, 66
Henkin, L. 165
Higman, G. 282
Hilbert, D. 128, 132
Horn, A. 145, 146, 168
Hurwitz, A. 99

Jónsson, B. 253, 255, 288

Kalicki, J. 300, 301, 303
Keisler, H. J. 170, 176
Kleene, S. C. 128, 133
Kochen, S.
Kuroš, A. G. 100, 254

Lambek, J. 217
Linnik, Yu. V. 100
Łos, J. 138
Lyndon, R. C. 144

Machover, M. 150
Mal'cev, A. I. 168, 217, 236, 303, 304
Marczewski, E. 241, 266
Morel, A. C. 148, 169, 174, 176
Morley, M. 209
Moufang, E. 100

Neumann, B. H. 106, 282, 304
Neumann, H. 282, 303, 304
Neumann, P. M. 282, 304
Novikov, P. S. 128

Parfenov, V. A. 304

Reichbach, M. 24
Riguet, J. 23
Robinson, A. 164, 203, 205, 208

Samuel, P. 87
Scott, D. S. 174, 176, 300, 303
Širšov, A. I. 100
Smirnov, D. M. 303
Soundarajan, T. 100
Swierczkowski, S. 251, 255

Tamura, T. 235, 236
Tarski, A. 138, 192, 253 255, 288, 303
Tietze, H. 222

Vaught, R. L. 192, 199
Vinogradov, A. A. 303

van der Waerden, B. L. 81

Zariski, O. 81
Zorn, M. 100

Subject Index

Abelian group 79
Absolute amalgamation 258
Absolutely closed formula 122
— directly indecomposable 61
— hereditary class 139
— quantifier-free 133
Absorption law 101, 107
Abstract class 163
— lattice 101
— property 36
Additive semilattice 101
Admissible sequence 178
— standard 178
Algebra 32
—, Boolean 5, 107
—, Cayley 98
—, $\mathfrak{K}$-attainable 236
—, $\mathfrak{K}$-indecomposable 236
—, linear, finite 94, 95
—, quaternion 95
—, relation 11
Algebraically closed field 88
— — — extension 88
Algebraic constant 266
— element 87
— field extension 87
— independence over field 89
Algebraic system(s) 32
— —, carrier of 32
— —, decomposable 57
— —, directly indecomposable 61
— —, elementarily equivalent 185
— —, equitype 32
— —, finitely approximable 60
— —, finitely presented 223
— —, finite type 32
— —, free, in a class 249
— —, indistinguishable 185
— —, locally embeddable 165
— —, mixed indecomposable 265
— —, order of 32
— —, partial 33
— —, prime 202
Algebraic system, quotient 45
— —, trivial 61
— —, type of 32
— —, union of 42
Algebra of subsets 5, 107, 178
Alphabet 109
α-complete filter 174
Alternative law 83
— ring 83
— skew field 91
Amalgamation (Amalgam) 256
Anticommutative ring 84
APC-formula 126
Arity 29, 32, 111
Associative groupoid 67
— law 57, 83, 103, 184
— ring 83
Associator 83
Atomic diagram 230
— formula 133
— variety, see minmal variety
Attainable class 236
— quasivariety 236
Automorphism of system 34
Axiomatic closure 128, 282
— quasirank 275
— rank 271
Axiomatizable class, finitely 163
Axiom of choice 21, 52
$\forall$-closure 189
$\forall$-formulas, prenex 129
$\forall$-subclass 137
$\forall\exists$-formula 140

Basic operation 32
— predicate 32
Basis 87
— for a system 223
— free in a class 249
— quasirank 275
— rank, of a system 266
— — of a variety 272
Binary operation 4, 29

Binary relations 8
— —, permutable 10
Birkhoff's theorem 267
Boolean algebra 5, 107
— lattice 107
— operations 5
— ring 8, 84
Bound variable 115

Cancellation semigroup 216
Canonical embedding 188
— homomorphism 237
— mapping 15
Cantor-Bernstein theorem 24
Cardinal (number) 24
— —, operations on 63ff.
Cardinality, of algebraic system 32
— of the continuum 24
Carrier 32
Cartesian power 29, 53
Cartesian product 8, 29, 52, 53
— —, complementary subproduct of 60
Categorical class of axioms 195
Categoricity 197
— in cardinality 199
— in signature 197
Category concept 56
Cayley algebra 98
Cayley number 97
— —, conjugate 99
Center 80
Chain 20
Chang-Morel theorem 169
Characteristic 86
— function 31
Closed formula 122
— in a system 37
Closure, axiomatic 128, 282
—, ∀- and ∃- 189
—, equivalence 18
—, formula 122
—, local 137
—, primitive 282
—, quasiprimitive 215
—, replete (replica-complete) 232
Cofinite 154
Collection of satisfiable formulas 163, 195
Commutative diagram 54
— group 79
— groupoid 67
— law 57
Commutative ring 84
Commutator 79
— subgroup 79
Compactness theorem 129, 163
Comparable elements 20
Compatible embedding 241
Complement 3, 106
Complementary product 60
Complemented lattice 106
Complete collection 196
Complete family, of congruences 58
— — of homomorphisms 54
Complete lattice 103
Completeness in signature 196
Complete pre-image 11
— regular product 179
Composition (composite) 9
—, free 237
—, injective 241, 258
—, $\mathfrak{K}$-free 237
— of systems with amalgamation 256
— of words 109
— with amalgamations 256
— homomorphism 237, 256
Concatenation 109
Conditional identity, *see* quasi-identity 149, 211
Conditionally filtered formula 160
— — predicate 160
Congruences 44
—, complete family of 58
—, improper (universal) 46
—, independent family of 59
—, kernel 46
—, trivial 46
—, verbal 235
Conjunction 111
— of relations 9
Consequence 196
—, formal 6
Consistent collection 128, 163, 196
Continuum problem 63
Countably complete filter 174
Cyclic semigroup 69

Decidable theory 183
Decomposition of system 57
Dedekind lattice 104
Dedekind's theorem 104
Defining set 211, 217
Derived subgroup 79
Diagram 102

Diagram, atomic 238
—, commutaive 54
— of a model 136
— of a system 197
Difunctional relation 22
Directly indecomposable 61
Direct power 53
Direct product 52
— —, associative law for 57
— —, commutaive law for 56
— —, proper 61
— —, weak 163
— spectrum, limit 42
— successor 21
Disjunction 111
— of relations 9
Distinguished element 32
Distributive lattice 104
— law 104
Divisible group 162
Division ring (skew field) 83, 84
Divisor of zero 82
Domain, partial multimapping 21
—, partial operation 29
Dual ordering 20
— prefix 142
Dyck's theorem 222

Element(s) complementary 106
—, direct successor 21
—, distinguished 32
—, generating 39
—, greatest 20
—, inverse 74, 82
—, invertible 78
—, maximal 20
— of infinite order 78
—, principal 32
— equivalence 185
— extension 188
— formula 143
— primitive subsystem 205
— submodel 187
— subsystem 187
Elementary embedding 187
— theory 128
— —, decidable 183
Embedding, canonical 188
—, compatible 241
—, elementary 187
—, isomorphic 35ff., 68
—, local 137
Embedding process 41
Embedment of words 110
Empty word 110
Endomorphism 56
Epimorphism 226
Equationally complete (I-complete) 295
— contained 270
— equivalent 270
Equitype systems 32
Equivalence relation 13
— — class 14
— —, fractional 49
— —, index of 59
— —, induced 14
— —, kernel 15
— closure 18
Equivalent formulas 127
Existential formula 130
Extension field 85ff.
∃-closure 189
∃-formulas, prenex 129
∃-reducible 201
∃∀-formula 140

Factor set, *see* quotient set
— system, *see* quotient system
False formula 118
Falsity 30
Field 84, 87ff., 200
Field adjunction 85
Field extension 85ff.
— —, algebraic 87
— —, algebraically closed 88
— —, finite 87
— —, purely transcendental 90
Filter 153
—, α-complete 174
—, countably complete 174
—, Fréchet 154
—, homogenous 173
—, principal, nonprincipal 153
Filtered power 157
— predicate 160
— product 157
— —, associative law for 184
— —, i-th power 157
Finite approximability 60
Finitely axiomatizable class 163
— definable variety 211
— presented system 223
Finite reduct 135
— system 32

Finite type 32
First isomorphism theorem 47
First- order language 126
— predicate 123
Formal language 109
Formula(s) 111, 115
—, absolutely closed 122
—, absolutely quantifier-free 133
—, applied predicate calculus (APC) 126
—, atomic 133
—, class of 186
—, closed 122
—, conditionally filtered 160
—, elementary 143
—, equivalent 127, 128
—, existential 130
—, false 118
—, Horn 145, 169
—, indentically true 127
—, multiplicatively stable 145
—, negative 144
— of signature σ 122
—, open 133
—, positive 143
—, prenex normal form 129
—, primitive 205
—, pure predicate calculus (PPC) 126
—, quasi-atomic 210
—, restricted predicate calculus (RPC) 126
—, satisfiable 127
—, Skolem 140
—, square stable 150
—, true 118
—, universal 130
—, value of 117
Fractional equivalence 49
Fréchet filter 154
Free algebraic system 247ff.
— basis 249
— composition 237
— —, associativity of 239
— factors 237
— product. 242, 256
— variable 115
Fully characteristic 291
Function 29
—, conditionally filtered 160
—, n-ary 29
—, partial 29
Function, *see* also operation
Function variable, *see* operator

Generating class 282
— elements 39
— set 39
— —, minimal 250
— system 283
Graph 31
Greatest element 20
Group 74
—, abelian (commutative) 79, 81
—, automorphism 78
—, commutator 79
—, divisible 162
—, locally solvable 80
—, metabelian 80
—, nilpotent 80
— of invertible elements 78
— of mappings 13
—, p^∞-type 79
—, symmetric 78
Groupoid 67
—, additive 67
—, associative 67
—, commutative 67
—, idempotent 67
—, multiplicative 67
— of quasivarieties 303
— of varieties 304

Henkin property 165
Hereditary class 139
Hilbert's algorithm 128
Homogeneous filter 173
Homomorphism(s) 34
—, canonical 237
—, complete family of 54, 56
—, composition 237
—, strong 45
Homomorphism theorem 45ff., 54ff., 113ff.
Horn formula 145, 169
— predicate 145
Horn's theorem 146

I-complete collection 295
Ideal 82
Idempotent groupoid 67
— law 101
Identically true formula 127
Identity (Identical relation) 149, 211
Identity element 67

I-equivalent 270
Identity relation 10
Image 11, 21
— of class 23
Immediate predecessor, successor 21
Independent family of congruences 59
— set 247
Index of equivalence 59
— — subgroup 77
— pair 70
Indistinguishability 185
Individual variable, *see* object variable 111
Induced equivalence 14
Inexact quasivariety 275
Injective composition 241, 258
Inner automorphism 76
Inverse element 74, 82
— law 74
— property 91
— relation 9
Invertible element 78
Isoclass 36
Isomorphic embedding 35ff., 60, 68ff., 78
Isomorphism 25, 34

Jacobi identity 84

Kernel congruence 46
— equivalence 15
$\mathfrak{K}$-attainable 236
$\mathfrak{K}$-congruence 235
$\mathfrak{K}$-free power 237
$\mathfrak{K}$-free composition 237, 256
$\mathfrak{K}$-free product 256
$\mathfrak{K}$-indecomposable 235
$\mathfrak{K}$-independent 247
$\mathfrak{K}$-morphism 228
$\mathfrak{K}$-replica 228

Lattice 100
—, abstract 101
—, Boolean 107
—, complemented 106
—, complete 103
—, distributive 104
—, modular 104
— of varieties 289ff.
— property 100
Left coset 76
Left division 72
Length of word 109
Lie ring 84
Like operations 32
Limit of direct spectrum 42
Lindenbaum's theorem 197
Linear algebra 94
Linearly ordered set 20, 199
Linear ordering 20
Local embedding 137
— family 40
Locally closed class 137
— finite family 285
Local submodel 136
Logical symbols 111
Loop, primitive 73
Lower central chain 80
— semilattice 100

Mal'cev's quasivariety theorem 213
Mapping 11, 34
—, composition 11
—, isomorphic 25
—, natural (canonical) 15, 23
—, partial 22
Maximal element 20
— filter, *see* ultrafilter
— ideal 86
Metabelian group 80, 245
Metalanguage 109
Metasymbol 109
Minimal generating set 250
— quasivariety 295
— subsystem 39
— variety 295, 303
Mixed free product 263
— indecomposable system 265
Model 32, 196
—, diagram of 136
—, prime 202
— representing algebraic system 33
Model complete 201
— completeness criterion 205
Modular lattice 104
— law 104
Monoid 69
Multimapping 21
Multiplicatively stable formula 145
— — predicate 145

n-ary function 29
— operation 29
— predicate 30
— relation 29
Natural mapping 15, 23

Negation 111
— of relation 9
Negative formula 144
Nilpotent group 80
Nonprincipal filter 153
Norm 96, 99
Normal subgroup 76
0-ary (Nullary) 31
Null variety 213

Object variable 111
Ω-projection, *see* Ω-reduct
Ω-reduct 135
Open formula 133
Operation(s) 29
—, basic 32
—, binary 4
—, like 32
—, n-ary 29
—, nullary 31
—, partial 29, 133
—, polar 304
—, unary 4
Operator domain 32
— variable (symbol) 111
Order of algebraic system 32
— of a type 32
— of group element 78
Ordering 19
— type 25
Ordinal(s) 25
—, comparable 27
—, operations on 28, 64

Partial algebraic system 33
— ordering 19
— mapping 22
— operation 29, 113
— —, n-ary 29
Partition 14
Permutable binary relations 10
Polarized class 304
— variety 304
Pole 304
Positive formula 143
Positive-formula predicate 143
PPC-formula 126
Predicate 30
—, basic 32
—, conditionally filtered 160
—, domain 32
—, filtered 160
—, first-order 123
Predicate, Horn 145
—, multiplicatively stable 145
—, n-ary 30
—, positive-formula 143
—, second-order 123
Predicate variable 111
— —, value of 117
Prefix 129
—, dual 142
Pre-image 11
Prenex formula 129
— normal form 129
Prime algebraic system 202
— model 202
— subfield 85
Primitive loop 73
— quasigroup 72
— subsystem 205
Principal filter 153
Principal subsystem 39
Product, cartesian 8, 29, 52, 53
—, complete regular 179
—, complementary 60
—, direct 52
—, mixed free 263
— of classes 303
— over amalgamated subsystem 256
—, regular 179
Projection 54, 60
Proper congruence 46
— direct product 61
Pseudoequivalence 16
Purely transcendental extension 90

Quasi-atomic formula 210
Quasicomplete family 54
Quasi-equationally complete (Q-complete) 295
— equivalent 271
Quasigroup, primitive 72
—, totally symmetric 73
Quasi-identity 149, 211
Quasiprimitive closure 215
Quasirank 275
Quasivariety 211
—, attainable 236
—, finitely presentable (definable) 211
—, inexact 275
—, minimal 295
Quaternion 95
Quaternion algebra 95
Q-complete collection 295
Q-equivalent 271

$Q_1 \dots Q_l$-formula 130
—, absolute 133
Quotient set 15, 22
Quotient system 45
— —, fully characteristic 291

Range, partial multimapping 21
Rank, axiomatic 271
—, basis 266, 272
— of identity 271
Reduced product, *see* filtered product
Reduct, finite 135
Refinement 264
Regular product 179
Relation(s) 8, 29
—, binary 8
—, composite 9
—, difunctional 22
—, equivalence 13
—, inverse 9
—, n-ary 29
—, negation of 9
—, permutable 10
—, stable 44
Relation algebra 11
Relational system, *see* model.
Replete (Replica-complete) class 232
Replica 228
Representing model 33, 37
Retract 258
Right coset 76
— division 72
Riguet's theorem 23
Ring 81
—, alternative 83
—, anticommutative 84
—, associative 83
—, Boolean 8, 84
—, commutative 84
—, division 83, 84
—, Lie 84
— with division 82
— with inverse property 91
— without zero divisors 82
Robinson's theorem 205
RPC-formula 126

Satisfiable formulas 127
— in a class 127
Second isomorphism theorem 49
Second-order language 109ff.
Second-order language, application 121ff.
Second-order predicate 123
— property 123
Semigroup(s) 67
—, cyclic 69
— of mappings 13
— of subsets 68
— of varieties 304
—, symmetric 67
— with cancellation 216
Sentence 122
Sequent 62
Set 1
— closed under operation 13
—, generating 39
—, linearly ordered 20, 199
—, partially ordered 19
—, well-ordered 21
— with lattice property 100
S-field. *See* skew field
Signature 114, 122
Skew field 83, 84
—, alternative 91
— of characteristic zero 86
—, quaternion 95
See also division ring
Skolem formula 140
Solvable, group, locally 80
Splitting sequence 178
Square stable formula 150
Stability 44, 130
—, multiplicative 145
—, square 150
Stable relation 44
Steinitz' theorem 90
Stone's theorem 107
Strong homomorphism 45
Structure, *see* lattice
— constants 95
Subalgebra 37
Subdecomposition 264
Subfield 85
—, prime 85
Subgroup 74
—, normal 76
— of finite index 77
Subgoupoid 67
Subloop 74
Submodel 37
—, elementary 187
—, local 136
Subproduct, complementary 60

Subquasigroup 72
Subring 81
Subsemigroup 67
Subskew field (Subdivision ring) 83
Subsystem 37
—, elementary 187
—, generated 39
—, minimal (principal) 39
—, primitive 205
Subword, initial 110
Symmetric difference 7
— group 78
— semigroup 67
System basis 223

Tarski-Łos theorem 138
Term, 111ff.
— in elements of set 114
—, polar 304
—, value of 112
Theorem, Birkhoff 267
—, Cantor-Bernstein 24
—, Chang-Morel 169
—, compactness (local) 129, 163
—, Dedekind 104
—, Dyck 222
—, existence 88
—, first isomorphism 47
—, homomorphism 45
—, Horn 146
—, Lindenbaum 197
—, Mal'cev quasivariety 213
—, Riguet 23
—, Robinson 205
—, second isomorphism 49
—, Steinitz 90
—, Stone 107
—, Tarski-Łos 138
—, Tietze 222
—, uniqueness 89
Theory, elementary 128, 183
Tietze's theorem 222
Totally symmetric quasigroup 73
Transcendence degree 90
Transfinite induction 21
Trivial algebraic system 61
— class 250
— congruence 46
— variety 213
True formula 118
— in class 127
Truth 30
Type of system 32

Ultraclosed class 163
Ultrafilter 155
Ultraproduct 162
Unary operation 4
Uniformly locally finite 285
Union of systems 42
Unit element 103
— variety 213
Universal congruence 46
— formula 130
Universally axiomatizable 137
Unoid 276
Upper semilattice 100
Υ-complete 196
Υ-description 189
Υ-equivalent algebras 186
Υ-extension 188
Υ-indistinguishability 186
Υ-reducible class 207
Υ-theory 186
$\Upsilon\Omega_1$-complete 199

Value, formula 117
—, predicate 117
—, term 112
Variety 211
—, finitely presented 212
—, generating system for 283
—, minimal (atomic) 295, 303
—, null 213
—, polarized 304
—, $\mathfrak{S}_{m,n}$ 303
—, trivial 213
—, unit 213
Vaught's test 199
Verbal congruence 235
Ξ-equivalent (Xi) 185

Weak direct product 163
Weakly complete 201
Well-ordered set 21
Well-ordering principle 21
Word(s) composition 109
—, embedment in 110
—, empty 110
— length 109

Zermelo's axiom 52
Zero divisors 82
— element, of groupoid 67
— — of lattice 103
Zorn's lemma 21

Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete

Eine Auswahl

97. Greub: Linear Algebra.
99. Cassels: An Introduction to the Geometry of Numbers.
104. Chung: Markov Chains with Stationary Transition Probabilities.
107. Köthe: Topologische lineare Räume I.
114. MacLane: Homology.
116. Hörmander: Linear Partial Differential Operators.
117. O'Meara: Introduction to Quadratic Forms.
120. Collatz: Funktionalanalysis und numerische Mathematik.
121./122. Dynkin: Markov Processes.
123. Yosida: Functional Analysis.
124. Morgenstern: Einführung in die Wahrscheinlichkeitsrechnung und mathematische Statistik.
125. Itô/McKean jr.: Diffusion Processes and their sample Paths.
126. Lehto/Virtanen: Quasikonforme Abbildungen.
127. Hermes: Enumerability, Decidability, Computability.
128. Braun/Koecher: Jordan-Algebren.
129. Nikodým: The Mathematical Apparatus for Quantum Theories.
130. Morrey jr.: Multiple Integrals in the Calculus of Variations.
131. Hirzebruch: Topological Methods in Algebraic Geometry.
132. Kato: Perturbation Theory for Linear Operators.
133. Haupt/Künneth: Geometrische Ordnungen.
134. Huppert: Endliche Gruppen I.
135. Handbook for Automatic Computation. Vol. 1/Part a: Rutishauser: Description of ALGOL 60.
136. Greub: Multilinear Algebra.
137. Handbook for Automatic Computation. Vol. 1/Part b: Grau/Hill/Langmaack: Translation of ALGOL 60.
138. Hahn: Stability of Motion.
139. Mathematische Hilfsmittel des Ingenieurs. 1. Teil.
140. Mathematische Hilfsmittel des Ingenieurs. 2. Teil.
141. Mathematische Hilfsmittel des Ingenieurs. 3. Teil.
142. Mathematische Hilfsmittel des Ingenieurs. 4. Teil.
143. Schur/Grunsky: Vorlesungen über Invariantentheorie.
144. Weil: Basic Number Theory.
145. Butzer/Berens: Semi-Groups of Operators and Approximation.
146. Treves: Locally Convex Spaces and Linear Partial Differential Equations.
147. Lamotke: Semisimpliziale algebraische Topologie.
148. Chandrasekharan: Introduction to Analytic Number Theory.
149. Sario/Oikawa: Capacity Functions.
150. Iosifescu/Theodorescu: Random Processes and Learning.
151. Mandl: Analytical Treatment of One-dimensional Markov Processes.
152. Hewitt/Ross: Abstract Harmonic Analysis. Vol. 2: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups.
153. Federer: Geometric Measure Theory.
154. Singer: Bases in Banach Spaces I.
155. Müller: Foundations of the Mathematical Theory of Electromagnetic Waves.
156. van der Waerden: Mathematical Statistics.

157. Prohorov/Rozanov: Probability Theory.
159. Köthe: Topological Vector Spaces I.
160. Agrest/Maksimov: Theory of Incomplete Cylindrical Functions and Their Applications.
161. Bhatia/Szegö: Stability Theory of Dynamical Systems.
162. Nevanlinna: Analytic Functions.
163. Stoer/Witzgall: Convexity and Optimization in Finite Dimensions I.
164. Sario/Nakai: Classification Theory of Riemann Surfaces.
165. Mitrinović/Vasić: Analytic Inequalities.
166. Grothendieck/Dieudonné: Eléments de Géométrie Algébrique I.
167. Chandrasekharan: Arithmetical Functions.
168. Palamodov: Linear Differential Operators with Constant Coefficients.
169. Rademacher: Topics in Analytic Number Theory.
170. Lions: Optimal Control of Systems Governed by Partial Differential Equations.
171. Singer: Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces.
172. Bühlmann: Mathematical Methods in Risk Theory.
173. F. Maeda/S. Maeda: Theory of Symmetric Lattices.
174. Stiefel/Scheifele: Linear and Regular Celestial Mechanics. Perturbed Two-body Motion—Numerical Methods—Canonical Theory.
175. Larsen: An Introduction of the Theory of Multipliers.
176. Grauert/Remmert: Analytische Stellenalgebren.
177. Flügge: Practical Quantum Mechanics I.
178. Flügge: Practical Quantum Mechanics II.
179. Giraud: Cohomologie non abélienne.
180. Landkof: Foundations of Modern Potential Theory.
181. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications I.
182. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications II.
183. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications III.
184. Rosenblatt: Markov Processes. Structure and Asymptotic Behavior.
185. Rubinowicz: Sommerfeldsche Polynommethode.
186. Wilkinson/Reinsch: Handbook for Automatic Computation II, Linear Algebra.
187. Siegel/Moser: Lectures on Celestial Mechanics.
188. Warner: Harmonic Analysis on Semi-Simple Lie Groups I.
189. Warner: Harmonic Analysis on Semi-Simple Lie Groups II.
190. Faith: Algebra: Rings, Modules, and Categories I.
191. Faith: Algebra: Rings, Modules, and Categories II.
192. Mal'cev: Algebraic Systems.
193. Pólya/Szegö: Problems and Theorems in Analysis I.
194. Igusa: Theta Functions.
195. Berberian: Baer $*$-Rings.
197. Benz: Geometrie der Algebren
200. Dold: Lectures on Algebraic Topology.